BEKG 2452
NUMERICAL METHODS
Solution of Linear Systems (Gauss Elimination)

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Lesson Outcome

Upon completion of this lesson, the student should be able to:

1. Reduce a matrix into row echelon form using row reduction method.
2. Solve a linear system by using Gauss Elimination and Gauss Elimination with partial pivoting.

Solution of **Linear Systems**, $A\mathbf{x} = \mathbf{b}$

Gauss Elimination

-Row Reduction
- GE with/without partial pivoting

LU Decomposition

- *A=LU*
- Apply forward and backward substitution

Gauss Seidel

- Strictly diagonal dominant matrix
- Run iteration

May be Exact or Approximated solution

3.1 Gauss Elimination

3.1.1 Elementary Row Operation

Aim to reduce a matrix into an upper triangular matrix.

I.e. $$\begin{bmatrix} \mathbf{X} & X & X \\ 0 & \mathbf{X} & X \\ 0 & 0 & \mathbf{X} \end{bmatrix}$$

X can be any different values

Pivot

3.1.1 Elementary Row Operation

1) Interchange any two rows

   - Interchange two rows i and j: $r_i \leftrightarrow r_j$

   E.g.

$$\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_2} \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$$

2) Multiply a row by a nonzero number

   - Replace row i by k times row j: $kr_i \rightarrow r_i$

   E.g.

$$\begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix} \xrightarrow{\frac{1}{2}r_1} \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$$

3.1.1 Elementary Row Operation

3) Row replacement operation:

- Replace row j by $mr_i + r_j$:

$$mr_i + r_j \to r_j$$

$m = -\frac{3}{1} = -3$

$m = -\frac{-2}{1} = 2$

$$\begin{bmatrix} 1 & 0 & 3 \\ 3 & 5 & -2 \\ -2 & 1 & -3 \end{bmatrix} \xrightarrow[2r_1+r_3]{-3r_1+r_2} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 5 & -11 \\ 0 & 1 & 3 \end{bmatrix}$$

$$\xrightarrow{-\frac{1}{5}r_2+r_3} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 5 & -11 \\ 0 & 0 & 26/5 \end{bmatrix}$$

$m = -\frac{1}{5}$

Example 3.1:

Reduce the matrix to an upper triangular matrix.

$$A = \begin{bmatrix} -1 & 2 & -5 \\ 2 & -1 & 6 \\ 1 & 1 & 3 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} -1 & 2 & -5 \\ 2 & -1 & 6 \\ 1 & 1 & 3 \end{bmatrix} \xrightarrow[r_1+r_3]{2r_1+r_2} \begin{bmatrix} -1 & 2 & -5 \\ 0 & 3 & -4 \\ 0 & 3 & -2 \end{bmatrix}$$

$$\xrightarrow{-r_2+r_3} \begin{bmatrix} -1 & 2 & -5 \\ 0 & 3 & -4 \\ 0 & 0 & 2 \end{bmatrix}$$

Example 3.2:

Reduce the following matrix to an upper triangular matrix.

$$A = \begin{bmatrix} 2 & 2 & 3 \\ -2 & 1 & 5 \\ 1 & 3 & 4 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 2 & 2 & 3 \\ -2 & 1 & 5 \\ 1 & 3 & 4 \end{bmatrix} \xrightarrow[\;]{\substack{r_1+r_2 \\ -\frac{1}{2}r_1+r_3}} \begin{bmatrix} 2 & 2 & 3 \\ 0 & 3 & 8 \\ 0 & 2 & 5/2 \end{bmatrix}$$

$$\xrightarrow{-\frac{2}{3}r_2+r_3} \begin{bmatrix} 2 & 2 & 3 \\ 0 & 3 & 8 \\ 0 & 0 & -17/6 \end{bmatrix}$$

3.1.2 Gauss Elimination Algorithm

Basic idea:

Solve the equivalent system which is in a simpler form compared to the original linear system.

Flow:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}$$

Linear System

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Matrix form

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$$

Augmented Matrix

Row Reduction

Backward Substitution

Example 3.3:

Solve the following linear system by using Gauss Elimination.

$$
\begin{aligned}
2x_1 + 3x_2 - x_3 &= 2 \\
4x_1 + 4x_2 - x_3 &= -1 \\
-2x_1 - 3x_2 + 4x_3 &= 1
\end{aligned}
$$

Solution:

Transform the linear system into matrix form:

$$
\begin{bmatrix} 2 & 3 & -1 \\ 4 & 4 & -1 \\ -2 & -3 & 4 \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
=
\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}
$$

Solution:

Augmented Matrix:

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & 2 \\ 4 & 4 & -1 & -1 \\ -2 & -3 & 4 & 1 \end{array}\right]$$

Row reduction:

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & 2 \\ 4 & 4 & -1 & -1 \\ -2 & -3 & 4 & 1 \end{array}\right] \xrightarrow[r_1+r_3]{-2r_1+r_2} \left[\begin{array}{ccc|c} 2 & 3 & -1 & 2 \\ 0 & -2 & 1 & -5 \\ 0 & 0 & 3 & 3 \end{array}\right]$$

Backward substitution:

$$3x_3 = 3 \implies x_3 = 1$$

$$-2x_2 + x_3 = -5$$

$$-2x_2 + (1) = -5 \implies x_2 = 3$$

$$2x_1 + 3x_2 - x_3 = 2$$

$$2x_1 + 3(3) - (1) = 2 \implies x_1 = 1$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \\ 1 \end{bmatrix}$$

Example 3.4:

Solve the following linear system by using Gauss Elimination.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 9 \\ 2x_1 - x_2 + x_3 &= 8 \\ 3x_1 \qquad\quad - x_3 &= 3 \end{aligned}$$

Solution:

Augmented Matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 2 & -1 & 1 & 8 \\ 3 & 0 & -1 & 3 \end{array}\right]$$

Solution (cont.):

Row reduction:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 2 & -1 & 1 & 8 \\ 3 & 0 & -1 & 3 \end{array}\right] \xrightarrow[]{\substack{-2r_1+r_2 \\ -3r_1+r_3}} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & -5 & -5 & -10 \\ 0 & -6 & -10 & -24 \end{array}\right]$$

$$\xrightarrow{-\frac{6}{5}r_2+r_3} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & -5 & -5 & -10 \\ 0 & 0 & -4 & -12 \end{array}\right]$$

Backward substitution:

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

3.1.3 Gauss Elimination with Partial Pivoting

Basic idea:

Avoid to have pivot element which is near to zero (it may cause to division by zero after normalization)

Flow:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}$$

Linear System

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$$

Augmented Matrix

Row Reduction

- **Pivot element is the largest absolute value** among entries underneath it
- Use only $r_i \leftrightarrow r_j$ and $mr_i + r_j \rightarrow r_j$

Backward Substitution

Example 3.5:

Solve the following linear system by using Gauss Elimination with partial pivoting.

$$
\begin{aligned}
0.3x_1 - 0.2x_2 + 10x_3 &= 71.4 \\
3x_1 - 0.1x_2 - 0.2x_3 &= 7.85 \\
0.1x_1 + 7x_2 - 0.3x_3 &= -19.3
\end{aligned}
$$

Solution:

Augmented matrix:

$$
\left[\begin{array}{ccc|c}
0.3 & -0.2 & 10 & 71.4 \\
3 & -0.1 & -0.2 & 7.85 \\
0.1 & 7 & -0.3 & -19.3
\end{array}\right]
$$

Solution (cont.):

Row reduction:

Max $\{|0.3|, |3|, |0.1|\} = 3$

$$\left[\begin{array}{ccc|c} \boxed{0.3} & -0.2 & 10 & 71.4 \\ \boxed{3} & -0.1 & -0.2 & 7.85 \\ \boxed{0.1} & 7 & -0.3 & -19.3 \end{array}\right]$$

$$\xrightarrow{r_1 \leftrightarrow r_2} \left[\begin{array}{ccc|c} \mathbf{3} & -0.1 & -0.2 & 7.85 \\ 0.3 & -0.2 & 10 & 71.4 \\ 0.1 & 7 & -0.3 & -19.3 \end{array}\right]$$

$$\xrightarrow[-0.0333r_1+r_3]{-0.1r_1+r_2} \left[\begin{array}{ccc|c} 3 & -0.1 & -0.2 & 7.85 \\ 0 & \boxed{-0.19} & 10.02 & 70.615 \\ 0 & \boxed{7.0033} & -0.2933 & -19.5617 \end{array}\right]$$

Max $\{|-0.19|, |7.0033|\} = 7.0033$

Solution (cont.):

$$\xrightarrow{r_1 \leftrightarrow r_2} \left[\begin{array}{ccc|c} 3 & -0.1 & -0.2 & 7.85 \\ 0 & \mathbf{7.0033} & -0.2933 & -19.5617 \\ 0 & -0.19 & 10.02 & 70.615 \end{array}\right]$$

$$\xrightarrow{0.0271 r_1 + r_3} \left[\begin{array}{ccc|c} 3 & -0.1 & -0.2 & 7.85 \\ 0 & 7.0033 & -0.2933 & -19.5617 \\ 0 & 0 & 10.1921 & 71.1451 \end{array}\right]$$

Backward substitution:

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2.9987 \\ -2.5007 \\ 6.9804 \end{bmatrix}$$

Example 3.6:

Solve the following linear system by using Gauss Elimination with partial pivoting.

$$
\begin{aligned}
2x_1 + 3x_2 - x_3 &= 2 \\
4x_1 + 4x_2 - x_3 &= -1 \\
-2x_1 - 3x_2 + 4x_3 &= 1
\end{aligned}
$$

Solution:

Augmented matrix:

$$
\left[\begin{array}{ccc|c}
2 & 3 & -1 & 2 \\
4 & 4 & -1 & -1 \\
-2 & -3 & 4 & 1
\end{array}\right]
$$

Solution:

Row reduction:

Max $\{|2|, |4|, |-2|\} = 4$

$$\left[\begin{array}{ccc|c} \boxed{2} & 3 & -1 & 2 \\ \boxed{4} & 4 & -1 & -1 \\ \boxed{-2} & -3 & 4 & 1 \end{array}\right] \xrightarrow{r_1 \leftrightarrow r_2} \left[\begin{array}{ccc|c} 4 & 4 & -1 & -1 \\ 2 & 3 & -1 & 2 \\ -2 & -3 & 4 & 1 \end{array}\right]$$

$$\xrightarrow[\frac{1}{2}r_1 + r_3]{-\frac{1}{2}r_1 + r_2} \left[\begin{array}{ccc|c} 4 & 4 & -1 & -1 \\ 0 & 1 & -1/2 & 5/2 \\ 0 & -1 & 7/2 & 1/2 \end{array}\right] \xrightarrow{r_2 + r_3} \left[\begin{array}{ccc|c} 4 & 4 & -1 & -1 \\ 0 & 1 & -1/2 & 5/2 \\ 0 & 0 & 3 & 3 \end{array}\right]$$

Backward substitution:

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \\ 1 \end{bmatrix}$$