BEKG 2452
NUMERICAL METHODS
Solution of Nonlinear Equations

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Lesson Outcome

Upon completion of this lesson, the student should be able to:

1. Identify the range that contains root(s).
2. Compute roots for nonlinear equations by using Bisection method, Simple Fixed-Point iteration and Newton-Raphson method.

Solution of a **Nonlinear Equation**, $f(x)=0$
(Polynomial, trigonometric, exponential, logarithmic equations)

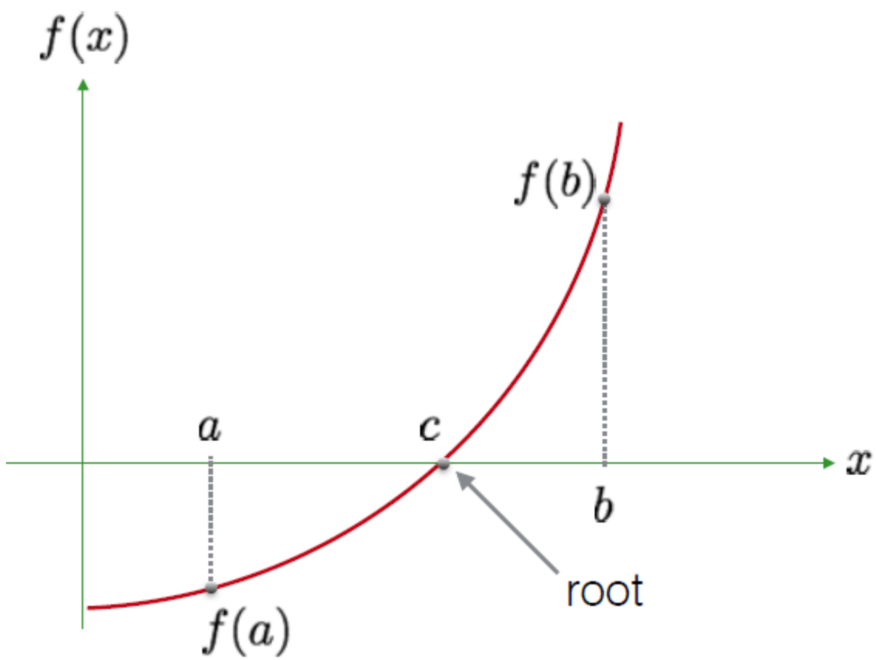
Bisection Method	Simple Fixed-Point Iteration	Newton-Raphson Method

Intermediate Value Theorem
- Find the **range** that contains a root (answer)

Start the iteration with respective algorithm to get the **approximation solution**

2.1 Intermediate Value Theorem

Let $f(x) = 0$ be a non-linear equation. If $f(x)$ is a continuous function and $\boldsymbol{f(a)f(b) < 0}$, then there exist at least a root in the interval (a, b).



When two points are connected by a curve:

- One point below x-axis
- One point above x-axis

Then there will be at least one root where the curve crosses the x-axis.

2.1 Intermediate Value Theorem

Example 2.1:

Given $f(x) = x^2 - 8x - 5$, use intermediate value theorem to find the interval that contains the negative root.

Solution:

$$f(0) = -5 < 0$$
$$f(-1) = 4 > 0$$
$$\therefore f(0)f(-1) < 0$$

Hence, the interval that contains the negative root is $(-1,0)$.

2.1 Intermediate Value Theorem

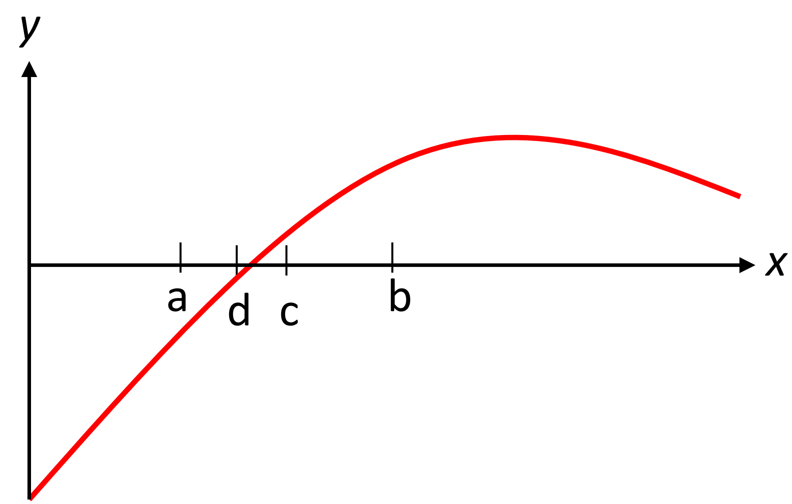
Exercise 2.1:

1) Use intermediate value theorem to find the interval that contains the root for $f(x) = x^3 + x + 3$.

2) Use intermediate value theorem to find the interval that contains the smallest positive root of $x = 2\sin x$.

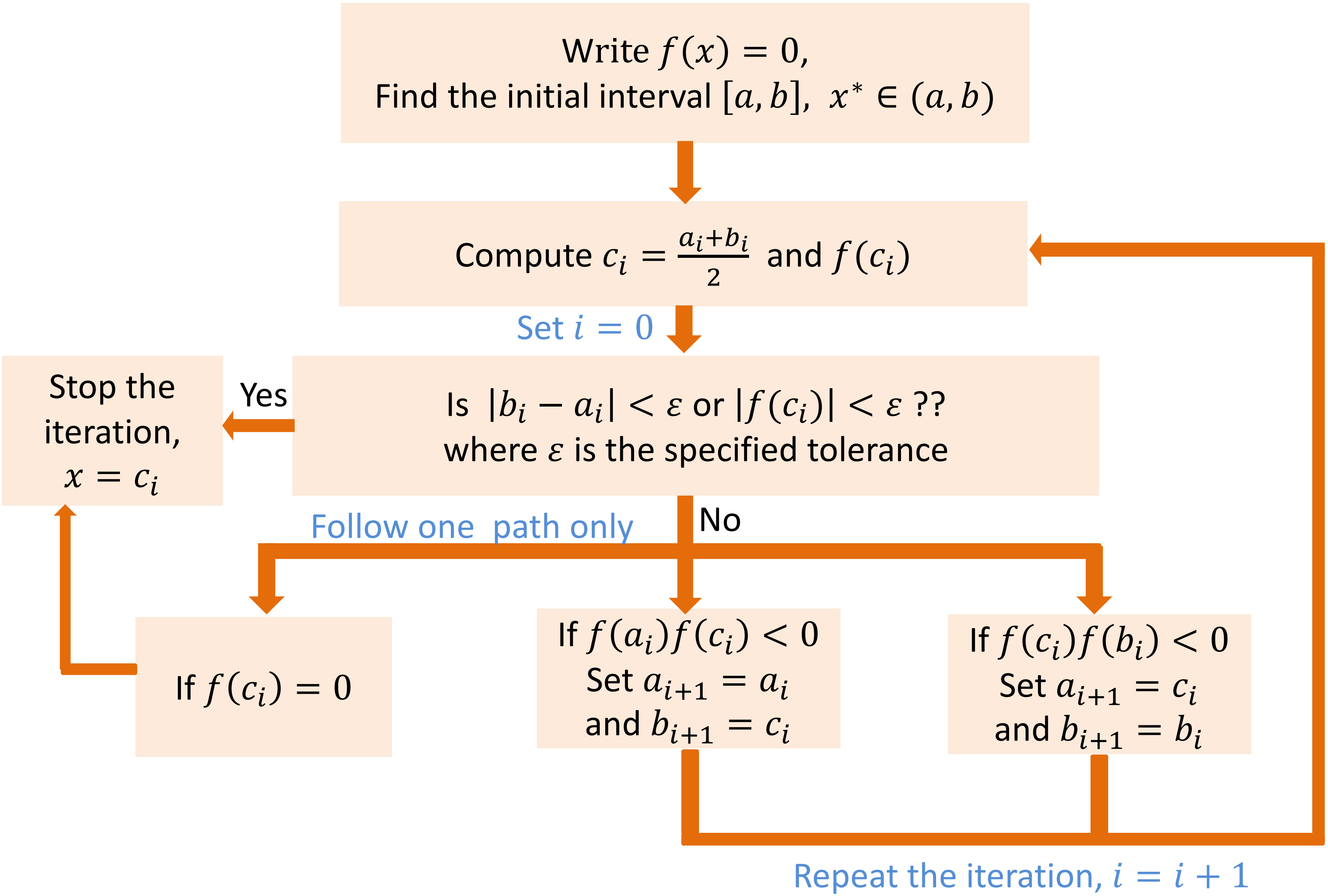
2.2 Bisection Method

The **bisection method** in mathematics is a root-finding method that **repeatedly bisects an interval** and then selects a subinterval that contains the root for further processing.



From (a,b), c is the midpoint of a and b
we choose (a,c), d is the midpoint of a and c
then we choose (d,c)
and so on…
until the range is small enough.

2.2.1 Bisection Method Algorithm



Example 2.2:

Find the root of $f(x) = x^2 - 3$ by using bisection method accurate to within $\varepsilon = 0.002$ and taking $(1,2)$ as starting interval.

(Answer correct to 4 decimal places)

Take that $|f(c_i)| < \varepsilon$ for your calculation.

Solution:

n	a_i	b_i	f(a_i)	f(b_i)	c_i	f(c_i)	lf(c_i)l
0	1.0000	2.0000	-2.0000	1.0000	1.5000	-0.7500	0.7500
1	1.5000	2.0000	-0.7500	1.0000	1.7500	0.0625	0.0625
2	1.5000	1.7500	-0.7500	0.0625	1.6250	-0.3594	0.3594
3	1.6250	1.7500	-0.3594	0.0625	1.6875	-0.1523	0.1523
4	1.6875	1.7500	-0.1523	0.0625	1.7188	-0.0459	0.0459
5	1.7188	1.7500	-0.0459	0.0625	1.7344	0.0081	0.0081
6	1.7188	1.7344	-0.0459	0.0081	1.7266	-0.0190	0.0190
7	1.7266	1.7344	-0.0190	0.0081	1.7305	-0.0055	0.0055
8	1.7305	1.7344	-0.0055	0.0081	**1.7324**	0.0013	0.0013

Root, $x = 1.7324$

Example 2.3

Using the bisection method, find the root of

$$f(x) = x^6 - x - 1$$

accurate to within $\varepsilon = 0.001$.

Given that $x_a = 1$ and $x_b = 2$.

Solution:

n	x_a	x_b	x_m	$f(x_a)$	$f(x_m)$	$\|x_b - x_m\|$
1	1.0000	2.0000	1.5000	-1	8.8906	0.5000
2	1.0000	1.5000	1.2500	-1	1.5647	0.2500
3	1.0000	1.2500	1.1250	-1	-0.0977	0.1250
4	1.1250	1.2500	1.1875	-0.0977	0.6167	0.0625
5	1.1250	1.1875	1.1562	-0.0977	0.2333	0.0312
6	1.1250	1.1562	1.1406	-0.0977	0.0616	0.0156
7	1.1250	1.1406	1.1328	-0.0977	-0.0196	0.0078
8	1.1328	1.1406	1.1367	-0.0197	0.0206	0.0039
9	1.1328	1.1367	1.1348	-0.0197	0.0004	0.0020
10	1.1328	1.1348	**1.1338**	-0.0197	-0.0096	0.0010

The root is, x **= 1.1338**

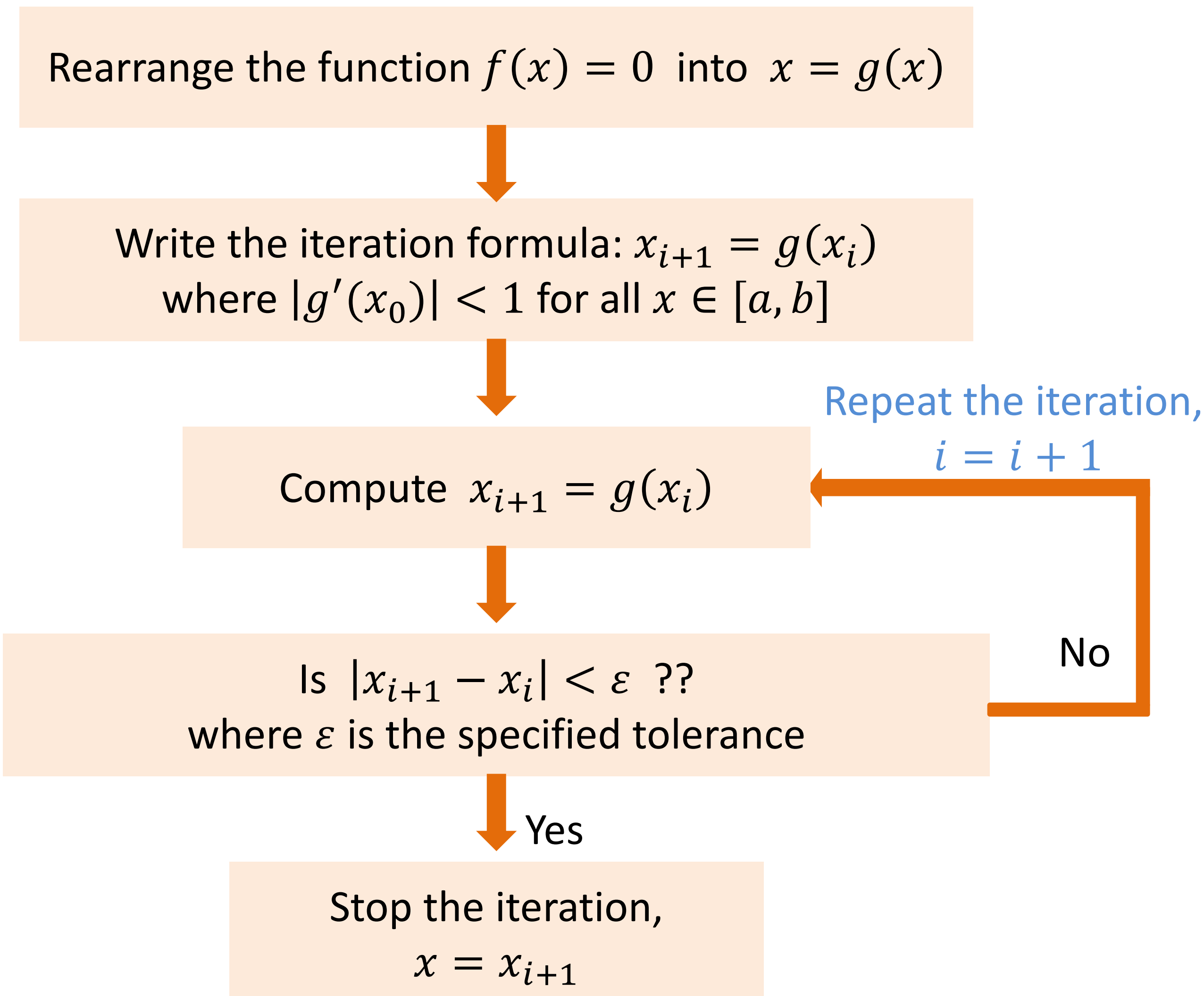
Exercise 2.2:

Find the root of $f(x) = e^x(3.2\sin x - 0.5\cos x)$ on the interval $[3,4]$ by using bisection method accurate to within $\varepsilon = 0.05$.

(Answer correct to 4 decimal places)

Take that $|f(c_i)| < \varepsilon$ for your calculation

2.3 Simple Fixed-Point Iteration



Remarks:
The Fixed-point iteration may **converge to a root different from the expected one**, or it **may diverge**. Different rearrangement will converge at different rates.

Example 2.4:

Given $f(x) = x^2 - 2x - 3$. Find the root of the function by using simple fixed-point method accurate to within $\varepsilon = 0.001$ and taking $x = 4$ as starting point.

(Answer correct to 4 decimal places)

Solution:

a) $x = g(x) = \sqrt{2x+3}$

$$g'(x) = \frac{1}{\sqrt{2x+3}} \text{ and } |g'(4)| = 0.3 < 1$$

This form will converge and give a solution

i	x_i	lx_{i+1}-x_il
0	4.0000	
1	3.3166	0.6834
2	3.1037	0.2129
3	3.0344	0.0694
4	3.0114	0.0229
5	3.0038	0.0076
6	3.0013	0.0025
7	3.0004	0.0008

The value converging to root of $x = 3.004$

Solution:

b) $x = g(x) = \frac{3}{x-2}$

$g'(x) = -\frac{3}{(x-2)^2}$ and $|g'(4)| = 0.75 < 1$

This form will converge and give a solution

i	x_i	lx_{i+1}-x_il
0	4.0000	
1	1.5000	2.5000
2	-6.0000	7.5000
3	-0.3750	5.6250
4	-1.2632	0.8882
5	-0.9194	0.3438
6	-1.0276	0.1083
7	-0.9909	0.0367
8	-1.0031	0.0122
9	-0.9990	0.0041

After 11 iterations, the value converging to root of $x = -1$

Solution:

c) $x = g(x) = \frac{x^2-3}{2}$

$g'(x) = x$ and $|g'(4)| = 4 > 1$

This form will diverge and give no solution

i	x_i	lx_{i+1}-x_il
0	4.0000	
1	6.5000	2.5000
2	19.6250	13.1250
3	191.0703	171.4453
4	18252.4322	18061.3618
5	166575638.3672	166557385.9350

value diverges

$g(x) = \frac{x^2-3}{2}$ is not a suitable form for simple fixed-point iteration

Example 2.5:

Find the root by using simple fixed-point iteration

$$f(x) = 3xe^{x} - 1$$

accurate to within $\varepsilon = 0.0001$. Assume $x_0 = 1$.

(Answer correct to 4 decimal places)

Solution:

There are two possible forms of $g(x)$:

$$x = g(x) = \frac{1}{3}e^{-x} \quad \text{and} \quad x = g(x) = \ln\left(\frac{1}{3x}\right)$$

$$g'(x) = -\frac{1}{3}e^{-x} \qquad\qquad g'(x) = -\frac{1}{x}$$

$$|g'(1)| = \mathbf{0.12} < \mathbf{1} \qquad\qquad |g'(0)| = \mathbf{1} \geq \mathbf{1}$$

Criteria is satisfied   Criteria is not satisfied

$$\therefore g(x) = \frac{1}{3}e^{-x}$$

Solution:

i	x_i	$\lvert x_i - x_{i+1} \rvert$
0	1.0000	
1	0.1226	0.8774
2	0.2949	0.1722
3	0.2482	0.0467
4	0.2601	0.0119
5	0.2570	0.0031
6	0.2578	0.0008
7	0.2576	0.0002
8	**0.2576**	0.0000

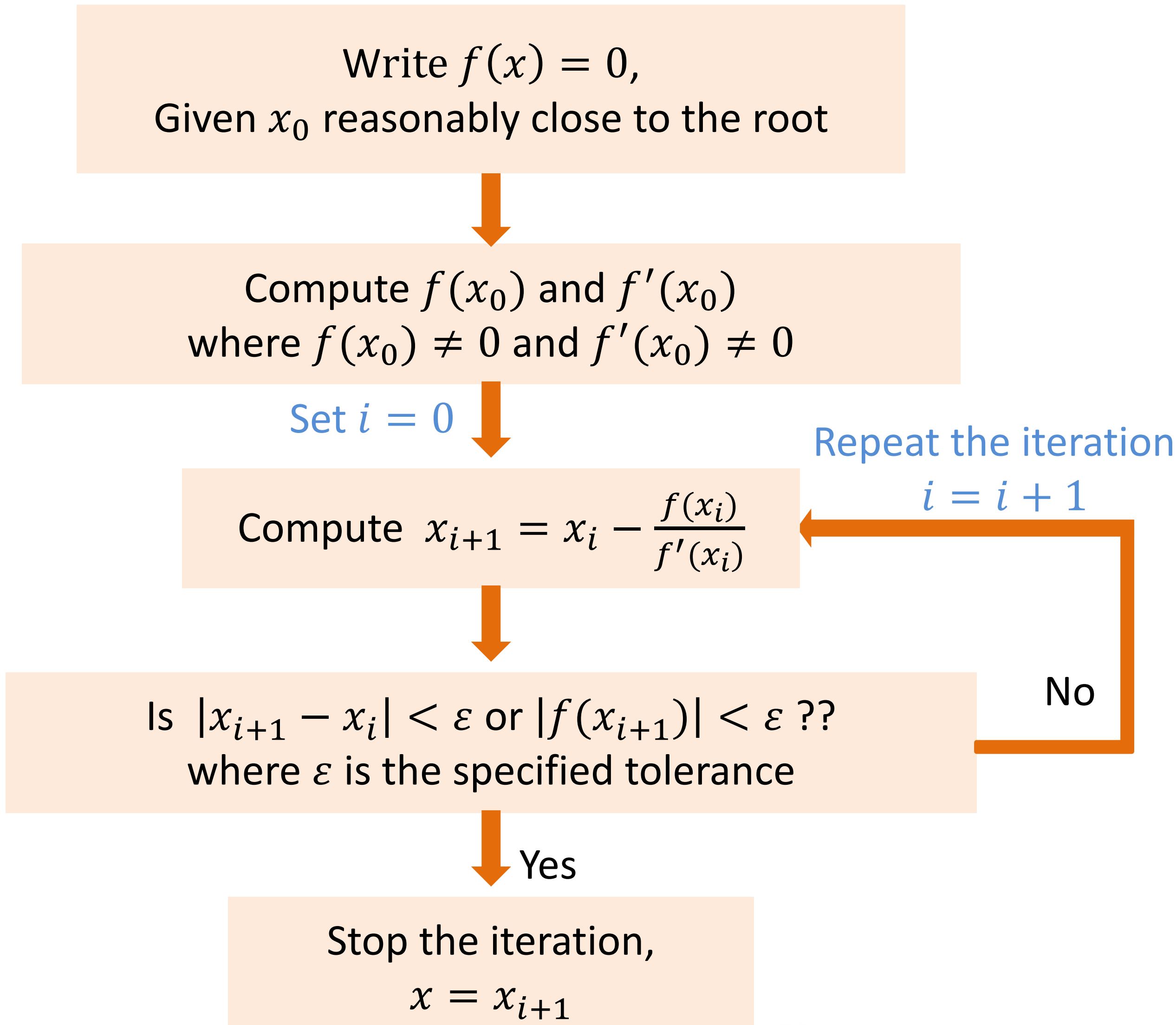
Thus, the root that satisfies the stopping criteria is $x = 0.2576$.

Exercise 2.3:

Locate the root of $f(x) = e^{-x} - x$ by using simple fixed-point iteration accurate to within $\varepsilon = 0.003$ where $x \in (0,1]$.

(Answer correct to 4 decimal places)

2.4 Newton-Raphson Method



Example 2.6:

Determine the root of the function

$$f(x) = e^x - \frac{2}{x}$$

by using Newton-Raphson method with $x_0 = 0.8$ accurate to within $\varepsilon = 0.0001$.

(Answer correct to 4 decimal places)

Take that $|f(x_i)| < \varepsilon$ for your calculation

Solution:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Given $f(x) = e^x - \frac{2}{x}$

hence, $f'(x) = e^x + \frac{2}{x^2}$

Check:

$f(0.8) = -0.2745 \neq 0$

$f'(0.8) = 5.3505 \neq 0$

i	x_i	f(x_i)	f'(x_i)	f(x_i)/f'(x_i)	|f(x_i)|
0	0.8000	-0.2745	5.3505	-0.0513	0.2745
1	0.8513	-0.0067	5.1024	-0.0013	0.0067
2	0.8526	-0.0000	5.0970	-0.0000	0.0000

Root, $x = 0.8526$

Reaching stopping criteria

Exercise 2.4:

Use the Newton-Raphson method to estimate the root of

$$f(x) = 3x + \sin x - e^x$$

starting from $x_0 = 0$ accurate to within $|x_{i+1} - x_i| \leq 0.0001.$

(Answer correct to 4 decimal places)