BEKG 2452
NUMERICAL METHODS
Solution of Linear Systems
(LU Decomposition)

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Lesson Outcome

Upon completion of this lesson, the student should be able to:

1. Decompose a matrix into a product of an upper and lower triangular matrices using LU Decomposition to solve a linear system.
2. Apply row reduction method to decompose a matrix.

Solution of **Linear Systems**, $A\mathbf{x} = \mathbf{b}$

Gauss Elimination

-Row Reduction
- GE with/without partial pivoting

LU Decomposition

- *A=LU*
- Apply forward and backward substitution

Gauss Seidel

- Strictly diagonal dominant matrix
- Run iteration

May be Exact or Approximated solution

3.2 LU Decomposition

Given a linear system,
$A\mathbf{x} = \mathbf{b}$

↓

Decompose $A = LU$

↓

$(LU)\mathbf{x} = \mathbf{b}$

↓

$L(U\mathbf{x}) = \mathbf{b}$

↓

Let $U\mathbf{x} = \mathbf{y}$, $\quad \therefore L\mathbf{y} = \mathbf{b}$

↓

solve $L\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$
(forward substitution)

↓

solve $U\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$
(backward substitution)

3.2 LU Decomposition

An $n \times n$ **nonsingular matrix** can be decomposed into a lower triangular matrix (L) and an upper triangular matrix (U) as follows:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$\boldsymbol{A} = \boldsymbol{L} \qquad \boldsymbol{U}$$

3.2.1 Steps of LU Decomposition

Step 1: Construct L and U

$$\boldsymbol{L\,U = A}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Multiply *L* and *U*:

$$\begin{bmatrix} \boxed{u_{11}} & \boxed{u_{12}} & \boxed{u_{13}} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix} = \begin{bmatrix} \boxed{a_{11}} & \boxed{a_{12}} & \boxed{a_{13}} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Compare each of the entries to obtain the values for *u* and *l*. (*i.e.* $u_{11} = a_{11}$)
(Sequence: $u_{11}, u_{12}, u_{13}, l_{21}, l_{31}, u_{22}, l_{32}, u_{23}, u_{33}$*)*

3.2.1 Steps of LU Decomposition

Step 2: Solve $L\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$ by using forward substitution

$$A\mathbf{x} = \mathbf{b}$$

When $A = LU$,

$$L(U\mathbf{x}) = \mathbf{b}$$

Let $U\mathbf{x} = \mathbf{y}$,

$$L\mathbf{y} = \mathbf{b}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Forward substitution

$$\begin{aligned} y_1 &= b_1 \\ l_{21}y_1 + y_2 &= b_2 \\ l_{31}y_1 + l_{32}y_2 + y_3 &= b_3 \end{aligned}$$

3.2.1 Steps of LU Decomposition

Step 3: Solve $U\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$ by using backward substitution

$$U\mathbf{x} = \mathbf{y}$$

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Backward substitution

$$\begin{aligned} u_{33}x_3 &= y_3 \\ u_{22}x_2 + u_{23}x_3 &= y_2 \\ u_{11}x_1 + u_{12}x_2 + u_{13}x_3 &= y_1 \end{aligned}$$

Example 3.7:

Solve the following linear system by using LU decomposition.

$$2x_1 + 3x_2 - x_3 = 2$$
$$4x_1 + 4x_2 - x_3 = -1$$
$$-2x_1 - 3x_2 + 4x_3 = 1$$

Solution:

Transform the linear system into matrix form:

$$\begin{bmatrix} 2 & 3 & -1 \\ 4 & 4 & -1 \\ -2 & -3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Solution (cont.):

Step 1: Construct L and U

$$\boldsymbol{L\,U = A}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 2 & 3 & -1 \\ 4 & 4 & -1 \\ -2 & -3 & 4 \end{bmatrix}$$

Multiply *L* and *U*:

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix} = \begin{bmatrix} 2 & 3 & -1 \\ 4 & 4 & -1 \\ -2 & -3 & 4 \end{bmatrix}$$

Compare each of the entries to obtain the values for *u* and *l*.

$$\therefore \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 2 & 3 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

Solution (cont.):

Step 2: Solve $L\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$ by using forward substitution

$$A\mathbf{x} = \mathbf{b},$$

when $A = LU$, $\quad (LU)\mathbf{x} = \mathbf{b} \quad \Rightarrow \quad L(U\mathbf{x}) = \mathbf{b}.$

Let $U\mathbf{x} = \mathbf{y}$, we have

$$L\mathbf{y} = \mathbf{b}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

By forward substitution:

$$y_1 = 2, \qquad \begin{aligned} 2y_1 + y_2 &= -1, \\ 2(2) + y_2 &= -1, \\ y_2 &= -5 \end{aligned} \qquad \begin{aligned} -y_1 + y_3 &= 1, \\ -(2) + y_3 &= 1, \\ y_3 &= 3 \end{aligned} \qquad \therefore \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix}$$

Solution (cont.):

Step 3: Solve $U\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$ by using backward substitution

$$U\mathbf{x} = \mathbf{y}$$

$$\begin{bmatrix} 2 & 3 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix}$$

By Backward substitution:

$$\begin{aligned} &3x_3 = 3, && -2x_2 + x_3 = -5, && 2x_1 + 3x_2 - x_3 = 2 \\ &x_3 = 1 && -2x_2 + (1) = -5, && 2x_1 + 3(3) - (1) = 2 \\ & && x_2 = 3 && x_1 = -3 \end{aligned}$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \\ 1 \end{bmatrix}$$

Alternate way to decompose A into LU

Row reduction:

Example 3.8:

$$A = \begin{bmatrix} \boxed{\begin{matrix} \mathbf{1} \\ 2 \\ 3 \end{matrix}} & \begin{matrix} 2 \\ -1 \\ 0 \end{matrix} & \begin{matrix} 3 \\ 1 \\ -1 \end{matrix} \end{bmatrix} \xrightarrow[-3r_1+r_3]{-2r_1+r_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & \boxed{\begin{matrix} \mathbf{-5} \\ -6 \end{matrix}} & \begin{matrix} -5 \\ -10 \end{matrix} \end{bmatrix} \xrightarrow{-\frac{6}{5}r_2+r_3} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -5 \\ 0 & 0 & \boxed{\mathbf{-4}} \end{bmatrix} = U$$

$\div 1$ $\quad\quad \div -5$ $\quad\quad \div -4$

$$\therefore L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 6/5 & 1 \end{bmatrix}$$

Alternate way to decompose A into LU

Row reduction:

Example 3.9:

$$A=\begin{bmatrix} \boxed{1} & -2 & -2 & -3\\ 3 & -9 & 0 & -9\\ -1 & 2 & 4 & 7\\ -3 & -6 & 26 & 2\end{bmatrix} \xrightarrow[3r_1+r_4]{\substack{-3r_1+r_2\\ r_1+r_3}} \begin{bmatrix} 1 & -2 & -2 & -3\\ 0 & \boxed{-3} & 6 & 0\\ 0 & 0 & 2 & 4\\ 0 & -12 & 20 & -7\end{bmatrix}$$

$$\div 1 \qquad\qquad \div -3$$

$$\xrightarrow{-4r_2+r_4} \begin{bmatrix} 1 & -2 & -2 & -3\\ 0 & -3 & 6 & 0\\ 0 & 0 & \boxed{2} & 4\\ 0 & 0 & -4 & -7\end{bmatrix} \xrightarrow{2r_3+r_4} \begin{bmatrix} 1 & -2 & -2 & -3\\ 0 & -3 & 6 & 0\\ 0 & 0 & 2 & 4\\ 0 & 0 & 0 & \boxed{1}\end{bmatrix} = U$$

$$\div 2 \qquad\qquad \div 1$$

$$\therefore L=\begin{bmatrix} 1 & 0 & 0 & 0\\ 3 & 1 & 0 & 0\\ -1 & 0 & 1 & 0\\ -3 & 4 & -2 & 1\end{bmatrix}$$

Example 3.10:

Solve the following linear system by using LU decomposition.

$$
\begin{aligned}
x_1 + 2x_2 + 3x_3 &= 9 \\
2x_1 - x_2 + x_3 &= 8 \\
3x_1 \qquad\quad - x_3 &= 3
\end{aligned}
$$

Solution:

Transform the linear system into matrix form:

$$
\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ 3 \end{bmatrix}
$$

Solution (cont.):

Step 1: Construct L and U

$$\boldsymbol{LU = A}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 0 & -1 \end{bmatrix}$$

Multiply *L* and *U*:

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 0 & -1 \end{bmatrix}$$

Compare each of the entries to obtain the values for *u* and *l*.

$$\therefore \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 6/5 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -5 \\ 0 & 0 & -4 \end{bmatrix}$$

Solution (cont.):

Step 2: Solve $L\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$ by using forward substitution

$$A\mathbf{x} = \mathbf{b},$$

when $A = LU$, $(LU)\mathbf{x} = \mathbf{b} \quad \Rightarrow \quad L(U\mathbf{x}) = \mathbf{b}.$

Let $U\mathbf{x} = \mathbf{y}$,

$$L\mathbf{y} = \mathbf{b}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 6/5 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ 3 \end{bmatrix}$$

By forward substitution:

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 9 \\ -10 \\ -12 \end{bmatrix}$$

Solution (cont.):

Step 3: Solve $U\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$ by using backward substitution

$$U\mathbf{x} = \mathbf{y}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -5 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ -10 \\ -12 \end{bmatrix}$$

By Backward substitution:

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

Example 3.11:

Solve the following linear system by using LU decomposition.

$$
\begin{aligned}
1.012x_1 - 2.132x_2 + 3.104x_3 &= 1.984 \\
-2.132x_1 + 4.096x_2 - 7.013x_3 &= -5.049 \\
3.104x_1 - 7.013x_2 + 0.014x_3 &= -3.895
\end{aligned}
$$

Solution:

Transform the linear system into matrix form:

$$
\begin{bmatrix} 1.012 & -2.132 & 3.104 \\ -2.132 & 4.096 & -7.013 \\ 3.104 & -7.013 & 0.014 \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
=
\begin{bmatrix} 1.984 \\ -5.049 \\ -3.895 \end{bmatrix}
$$

Solution (cont.):

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -2.1067 & 1 & 0 \\ 3.0672 & 1.1978 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 1.012 & -2.132 & 3.104 \\ 0 & -0.3955 & -0.4738 \\ 0 & 0 & -8.9391 \end{bmatrix}$$

By forward substitution:

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1.984 \\ -0.8693 \\ -8.9391 \end{bmatrix}$$

By backward substitution:

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Hands-on:

Solve the following linear system by using LU decomposition.

$$4x_1 = 3 + x_2 - x_3$$
$$-x_1 = 2 - 7x_2 - 3x_3$$
$$x_1 = 1 - 3x_2 - 5x_3$$

Answer:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.9767 \\ 0.5698 \\ -0.3372 \end{bmatrix}$$