INTRODUCTION TO MECHANICAL ENGINEERING
BMCG 2423
MECHANICS : TORSION

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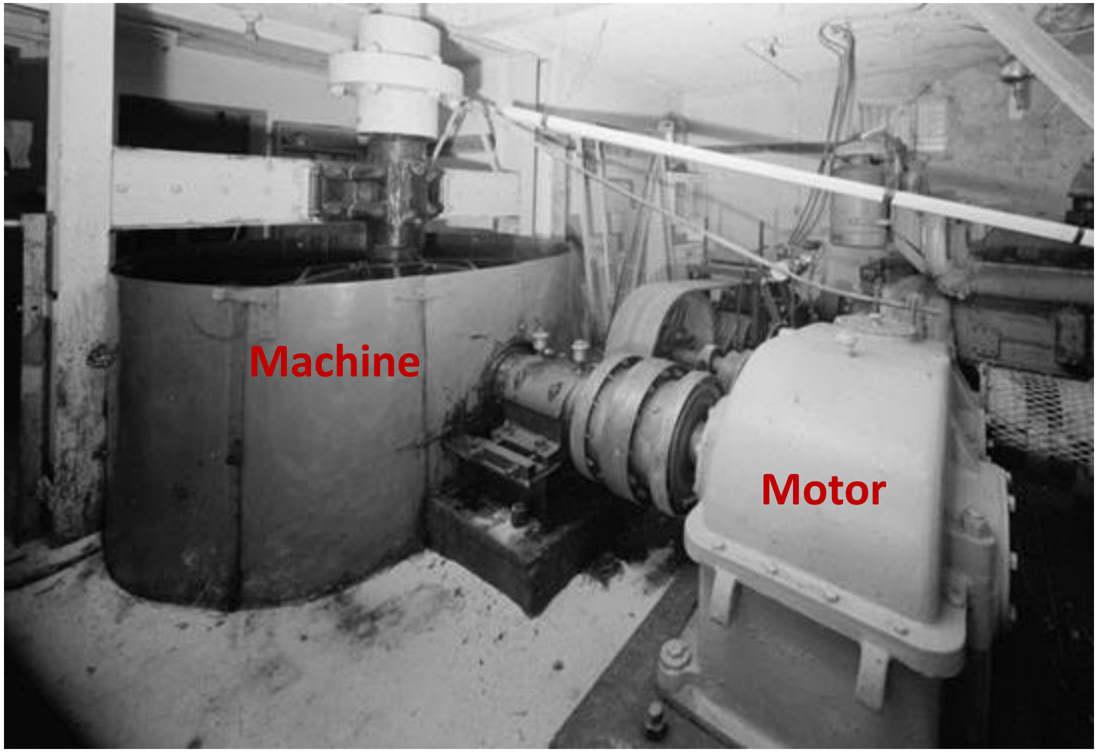
Lesson Outcome

At the end of lesson, students will be able to:

- Determine the shear stresses in a circular shaft due to torsion.
- Determine the angle of twist.

Application

Can the shaft transmit power from the motor to machine wthout failure?



Torsional loads on circular shaft

- Two important parameters need to consider when a shaft subjected to twisting couple or torques are:

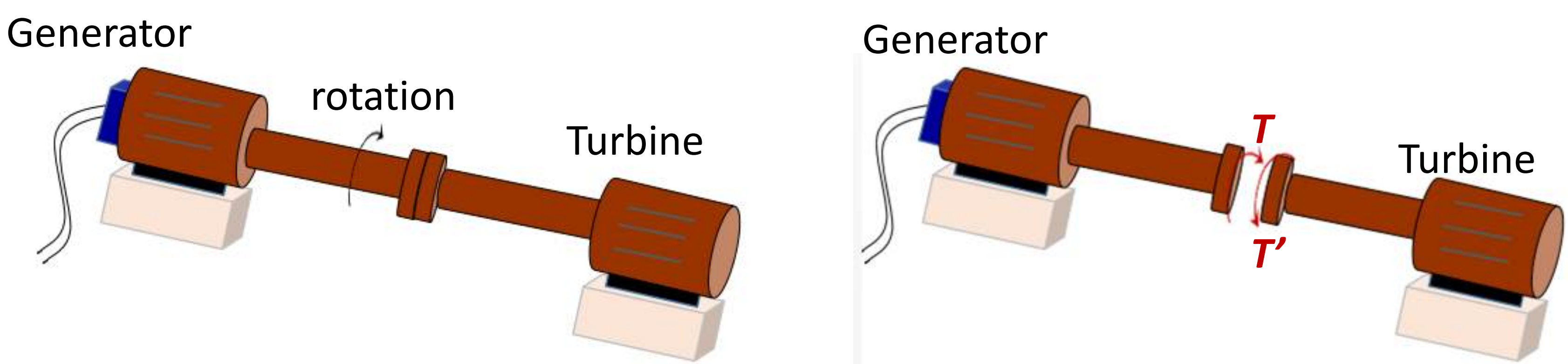
  i) Stress

  ii) Strain

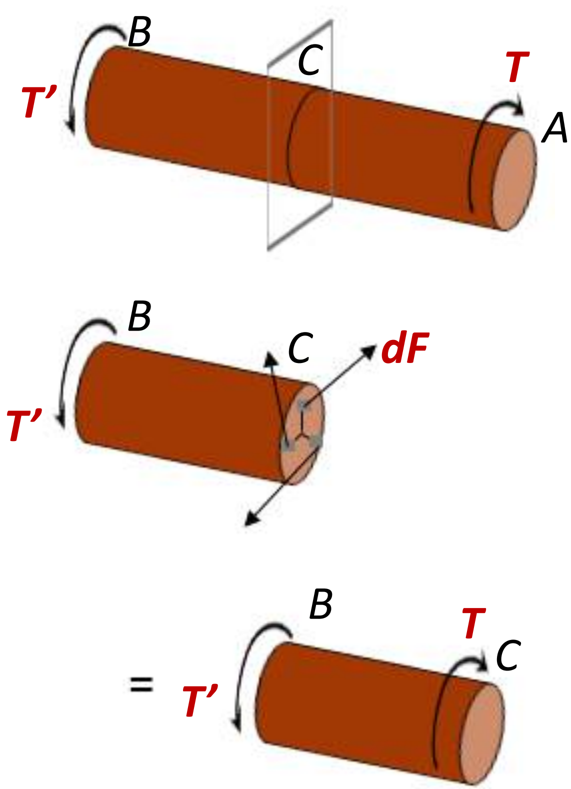
- It will determine whether the shaft will fail or not.

Torsional loads on circular shaft

- Consider a system consist of one turbine and generator below:
- Shaft will transmits power from turbine to generator.
- On the coupling or shaft cross sections, there are equal and opposite direction of torque generated.



Net Torque due to Internal Stress

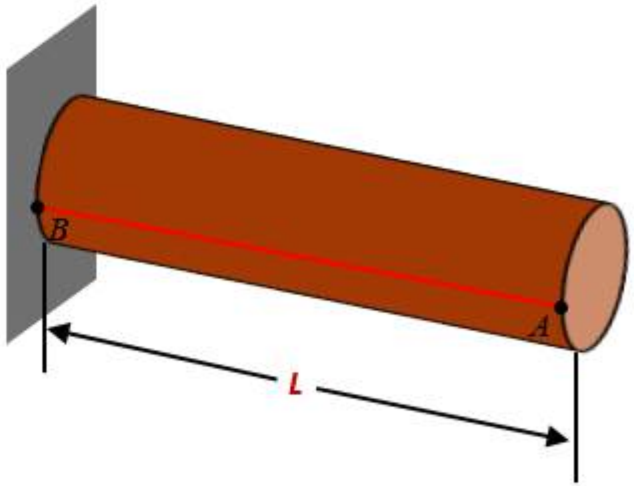


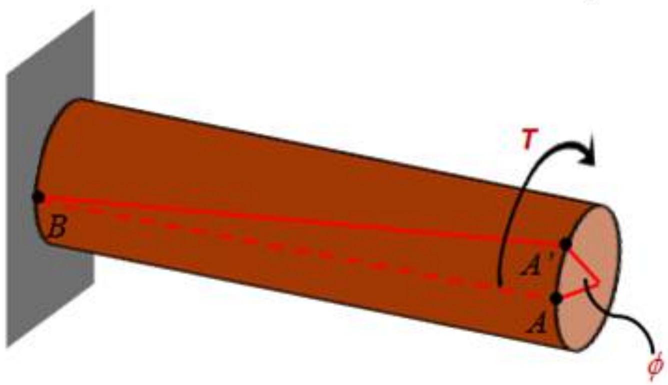
- The magnitude of internel shering stress is equal and opposite to the applied torque.

$$T = \int \rho dF = \int \rho(\tau dA)$$

- Distribution of sherring strees is statically indeterminate. Thus it must must consider the shaft defromation.
- The distribution of shearing stresses due to torsional loads cannot be assumed uniform unlike normal stress due to axial load.

Deformation of shaft





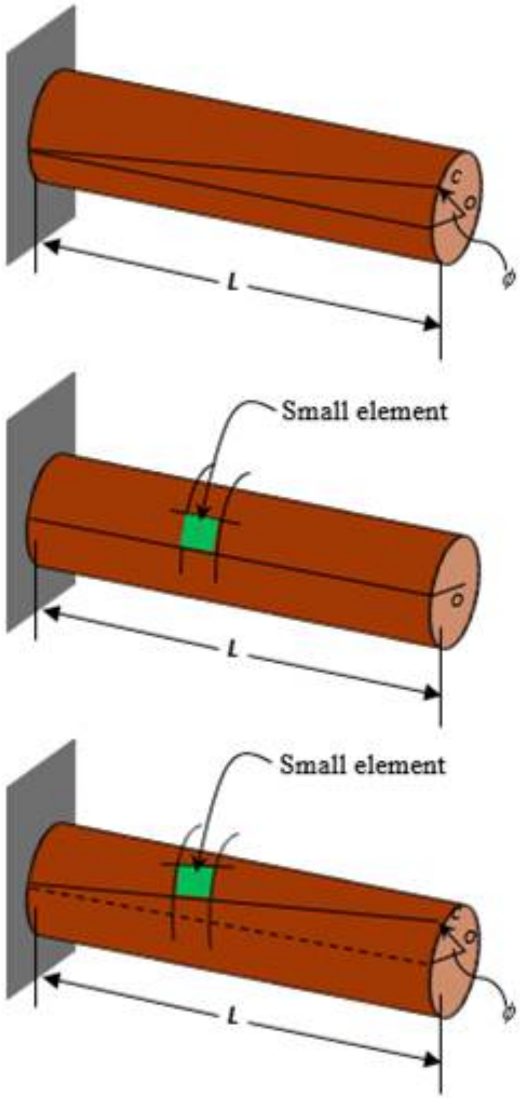
- The angle of twist for shaft is proportional to applied torque and length of shaft.

  $$\emptyset \propto T$$

  $$\emptyset \propto L$$

- When torsional load applied on shaft, every every cross sectional are:
  - Remain plane and distorted when axissymetric (for example circular cross section.
  - Distorted when non-axissymetric (for example non-circular shaft).

Shearing Strain



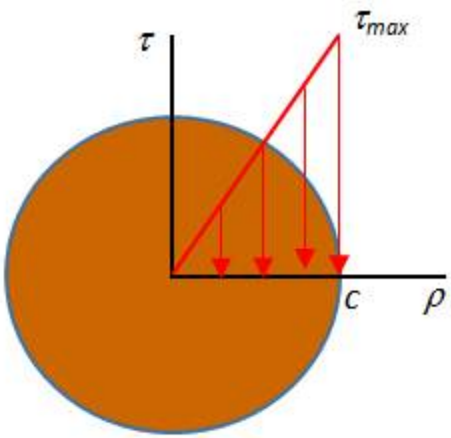
- When a circular shaft subjected to trosional load T, an elemnet on the interior cyclinder will deforms into a rhombus.
- Since the end of element remain planar, the shear strain is equal to angle of twist and follow that:

$$L\gamma = \rho\phi \quad \text{or} \quad \gamma = \frac{\rho\phi}{L}$$

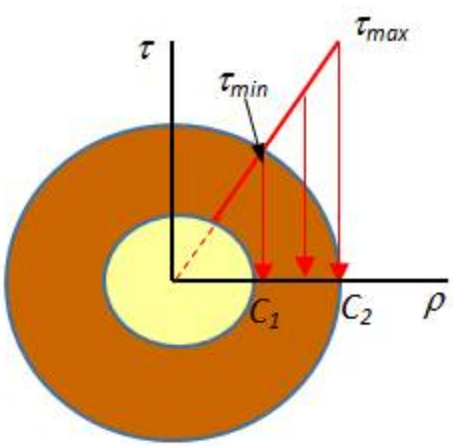
- The shear strain is proportional to angle of twist and radius of shaft.

$$\gamma_{max} = \frac{c\phi}{L} \quad \text{and} \quad \gamma = \frac{\rho}{c}\gamma_{max}$$

Stress in Elastic Range



$$J = \frac{1}{2}\pi c^4$$



$$J = \frac{1}{2}\pi({c_2}^4 - {c_1}^4)$$

- By multiplying the previous equation with shear modulus,

$$G\gamma = G\,\frac{\rho}{c}\gamma_{max}$$

and from Hooke's Law, $\tau = G\gamma$

$$\tau = \frac{\rho}{c}\tau_{max}$$

The shesing stress varies linearly with radius of the section.

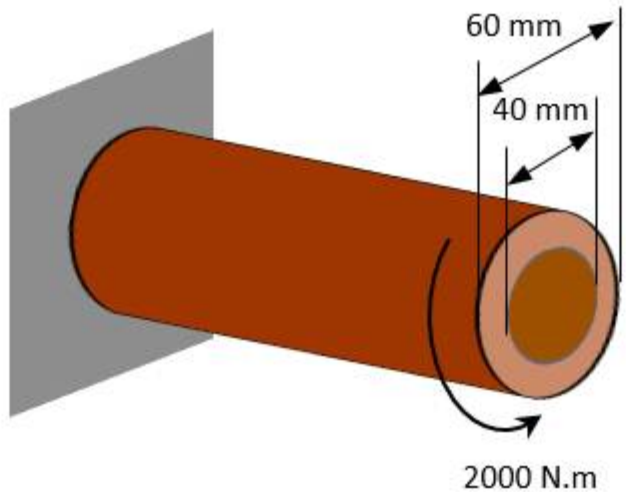
- The sum of moments is equal to the torque at the section of shaft.

$$T = \int \rho\tau dA = \frac{\tau_{max}}{c}\int \rho^2 dA = \tau_{max}\frac{J}{c}$$

- The elastic torsion formula:

$$\tau_{max} = \frac{Tc}{J} \quad \text{and } \tau = \frac{T\rho}{J}$$

Example 1



For the hollow and loading shaft shown, determine:

a) Maximum shearing stress
b) Diamater of shaft if the hollow shaft is change to solid shaft and their maximum shearing stress is the same as part (a)

Solution

$$c_1 = \frac{d_1}{2} = \frac{40}{2} = 20\ mm \qquad c_2 = \frac{d_2}{2} = \frac{60}{2} = 30\ mm$$

$$J = \frac{\pi}{2}(c_2{}^4 - c_1{}^4) \qquad = \frac{\pi}{2}(0.03^4 - 0.02^4) \quad = 1.021 \times 10^{-6}\ m^4$$

(a) $\tau_{max} = \frac{Tc}{J} = \frac{(2000)(0.03)}{1.021x10^{-6}}$ = **58.76 MPa**

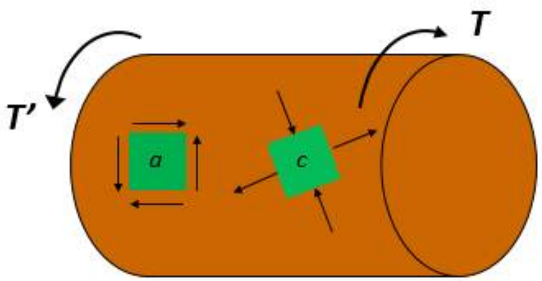
(b) $\tau = \frac{Tc}{J}$; $J = \frac{\pi}{2}c^4$; $\tau = \frac{2T}{\pi c^3}$

$$c^3 = \frac{2(2000)}{\pi(58.76\ x\ 10^6)} = 21.67 \times 10^{-6}\ m^3$$

c = 0.0279 m

diameter, d = **0.0558 m**

Torsional Failure Modes

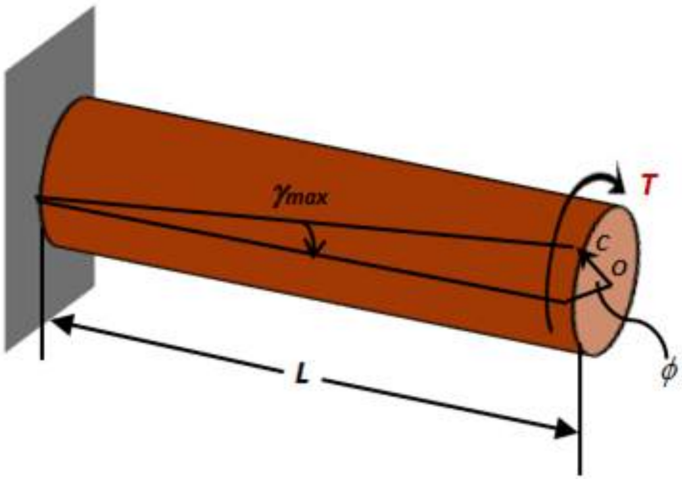


$$\tau_{max} = \frac{Tc}{J} \qquad \sigma_{45^\circ} = \pm\frac{Tc}{J}$$

- Ductile material normally stronger in tension campared to brittle material.
- The ductile material normally fail in shearing mode.

- Ductile material that subjected to torsional load will breaks along a plane of maximum shear (perpendicular to shaft axis).
- Whereas, the brittle specimen subjected to torsional load will breaks a plane at which tension is maximum (45° to shaft axis).

Angle of Twist in Elastic Range

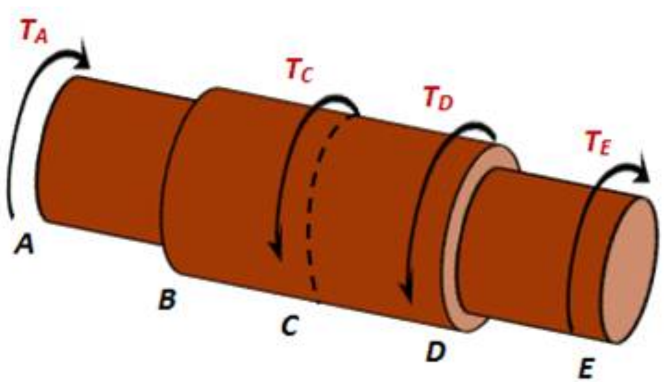


- The relationship between angle of twist and maximum shearing stress:

$$\gamma_{max} = \frac{c\phi}{L}$$

- By applying Hooke's Law in the elastic range,

$$\gamma_{\max} = \frac{\tau_{max}}{G} = \frac{Tc}{JG}$$



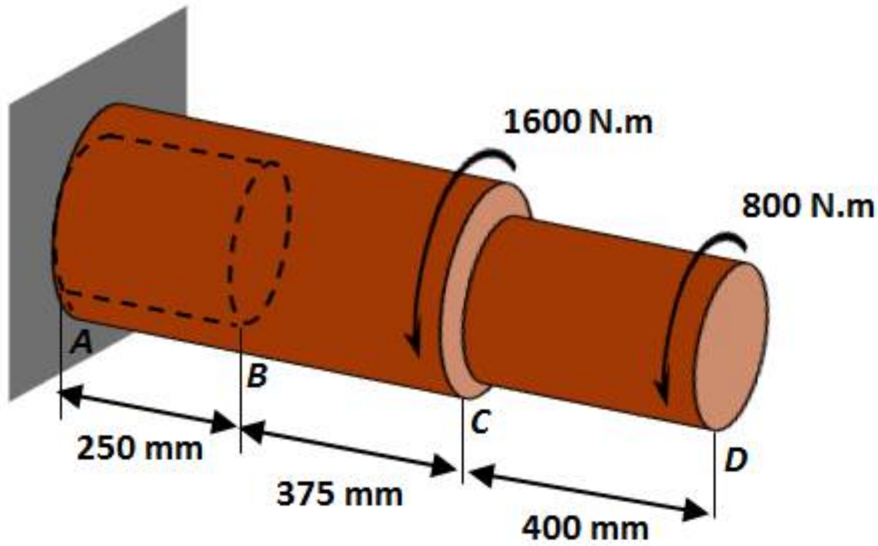
- Equating that both eaquation for angle of twist,

$$\phi = \frac{TL}{JG}$$

- Angle of twist for different corss section of shaft,

$$\phi = \sum_i \frac{T_i L_i}{J_i G_i}$$

Example 2



Rod *ABC* is made of brass (*G*= 39 GPa) with outer diameter of 60 mm. The portion rod *AB* is hollow has an dinner diameter of 40 mm. The rod *ABC* is bonded to aluminum rod *CD* (*G*= 27 GPa) with diameter of 36 mm. Determine the angle of twist at end *D*.

Solution

Portion CD

$G = 27$ GPa ; $L = 0.400$ m ; $T = 800$ N.m ;

$C = d/2 = 0.018$ m

$$J = \frac{\pi}{2}C^4 = \frac{\pi}{2}(0.018)^4 = 164.896\, x\, 10^{-9} m^4$$

$$\phi_{C/D} = \frac{TL}{JG} = \frac{(800)(0.400)}{(27\, x\, 10^9)(164.896\, x\, 10^{-9})} = 71.875\, x\, 10^{-3} \text{ rad}$$

Portion BC

$G = 39$ GPa ; $L = 0.375$ m ; $C = d/2 = 0.030$ m

$T = 800 + 1600 = 2400$ N.m

$$J = \frac{\pi}{2}C^4 = \frac{\pi}{2}(0.030)^4 = 1.27234\, x\, 10^{-6} m^4$$

$$\phi_{B/C} = \frac{TL}{JG} = \frac{(2400)(0.375)}{(39\, x\, 10^9)(1.27234\, x\, 10^{-6})} = 18.137 x\, 10^{-3} \text{ rad}$$

Portion AB

$G = 39$ GPa ; $L = 0.250$ m ;

$C_1 = d_1/2 = 0.020$ m $C_2 = d_2/2 = 0.030$ m

$T = 800 + 1600 = 2400$ N.m

$$J = \frac{\pi}{2}(C_2^4 - C_1^4)$$

$$= \frac{\pi}{2}(0.030^4 - 0.020^4) = 1.02102 x 10^{-6}\ m^4$$

$$\phi_{A/B} = \frac{TL}{JG} = \frac{(2400)(0.250)}{(39\ x\ 10^9)(1.02102\ x\ 10^{-6})} = 15.068 x\ 10^{-3} \text{ rad}$$

Angle of twist at *D*,

$\phi_D = \phi_{C/D} = \phi_{B/C} = \phi_{A/B}$

= 105.080 x 10^{-3} rad

Design of Transmission Shafts

- Shafts normally used to transmit power from source (motor) to a machine.
- There are two important principles in design shaft specicification:
  - Power
  - Speed
- The selection of material and their size need to meet the performance specification without exceeding the maximum shear stress allowed.
- Power, ***P*** is defined as work perfomed per unit time.

Design of Transmission Shafts

- The power for a rotating shaft with a torque is define as,

  $$P = T\omega,$$

  where shaft angular velocity $\omega = d\theta/dt$.

- Since one cycle = 2π rad, $\omega = 2\pi f$. Therefore,

  $$P = 2\pi f T$$

- Considering shaft cross-section in design,

  $$\tau_{max} = \frac{Tc}{J}$$

  Where,

  $$\frac{J}{c} = \frac{\pi}{2}c^3 = \frac{T}{\tau_{\max}} \quad \text{(solid shafts)}$$

  $$\frac{J}{c_2} = \frac{\pi}{2c_2}\left(c_2^4 - c_1^4\right) = \frac{T}{\tau_{\max}} \quad \text{(hollow shafts)}$$

Example 3

An engineer need to design a steel shaft 2.5 meter length, which is use to transmit a power of 150 kW at 360 rpm speed. In his design consideration, the maximum stress is not allowed more than 50 MPa and the angle of twist is not exceed 3°. Using G=77.2 GPa, determine the minimum diameter size of the shaft.

Solution

$P = 150\ kW = 150\ x\ 103\ W$; $\quad f = \dfrac{360}{60} = 6\ Hz$

$$T = \frac{P}{2\pi f} = \frac{150\ x\ 10^3}{6} = 3.9789\ x\ 10^3\ N.m.$$

Stress requirement = 50 MPa

Knowing that, $\tau = \dfrac{Tc}{J} = \dfrac{2T}{\pi C}$

$$c = \sqrt[3]{\frac{2T}{\pi\tau}} = \sqrt[3]{\frac{2(3.9789\ x\ 10^3)}{\pi(50x10^3)}}$$

$$= 37.0\ x\ 10^{-3} m$$

= 37 mm

Angle of twist requirement, $\phi = 3° = 52.36 \times 10^{-3}$ rad

$$\phi = \frac{TL}{JG} = \frac{2TL}{\pi G c^4}$$

$$c = \sqrt[4]{\frac{2TL}{\pi G \phi}} = \sqrt[4]{\frac{2(3.9789\,x\,10^3)(2.5)}{\pi(77.2x10^9)(52.36\,x\,10^{-3})}}$$

$= 35.38 \times 10^{-3}$ m

$= 35.38$ mm

Use the larger value,

$c = 37.0$ mm

$\boldsymbol{d = 2c = 74}$ **mm**

End of Lesson

Recall:

- What are two important parameters that need to be considered when a shaft is subjected to torques ?
- What happen when torsional load is applied on circular shaft?
- Can you derive the elastic torsion formula?
- Describe the torsional failure modes?
- State the relationship between angle of twist and maximum shearing stress?

References

- Beer, F.P., Johnston Jr., E.R., DeWolf, J.T, 2014, **Mechanics of Materials**, Fourth Edition in SI Units, McGrawHill, Singapore.