CALCULUS FOR TECHNOLOGY (BETU 1023)

WEEK 7

APPLICATIONS OF DIFFERENTIATION

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- OPTIMIZATION PROBLEMS

LEARNING OUTCOMES

At the end of this topic, students should be able to:

- Understand the definition of optimization
- Distinguish between maximum or minimum problem
- Solve application problems using absolute value method
- Solve application problems using second derivative test

OPTIMIZATION??

- The process of finding maximum or minimum values is called optimization.
- In optimization problems we are looking for the largest (maximum) value or smallest (minimum) value.
- In order to determine the function is maximum or minimum we can use Absolute Value or Second Derivative Test.

STEPS IN SOLVING OPTIMIZATION PROBLEMS

- Understand the problem
- Draw a diagram (if needed)
- Introduce notation
- Express the equation with one unknown (variable) only which is the equation to be maximized or minimized
- Used the suitable method to find maximum or minimum value
  - Method of absolute value
  - Second derivative test

Method of Absolute Value??

- Find the values of equation at the endpoints of the interval (only applied for closed interval).
- Find the values of equation at the critical numbers of equation where:
  - Set up the first derivative of equation is equal to zero
- Maximum value when the value in step 1 and 2 above give the largest value.
- Minimum value when the value in step 1 and step 2 above give the smallest value.

Second Derivative Test??

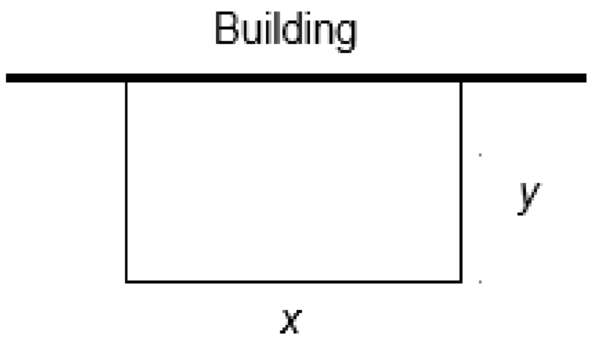
- Find the values of equation at the critical numbers of equation where:
  - Set up the first derivative of equation is equal to zero
- Maximum value when the second derivative for the critical numbers is negative value.
- Minimum value when the second derivative for the critical numbers is positive value.

EXAMPLE 1

We need to enclose a field with a fence. We have 500 feet of fencing material and a building is on one side of the field and so won't need any fencing. Determine the dimensions of the field that will enclose the largest area?

SOLUTION 1

- To find the dimensions (length and width) of the field that give maximum area.
- Let
  - x = length of the field
  - y = width of the field



SOLUTION 1

- Express the equation of area since we want to find the dimensions of maximum area.
- Area = length x width

$$A = xy$$

$$500 = x + 2y$$

$$x = 500 - 2y$$

$$A = y(500 - 2y)$$

SOLUTION 1

- Solve using the method of absolute value
- Note that $y \geq 0$ and $y \leq 250$, so the function we wish to maximize is

$$A(y) = y(500 - 2y), \quad 0 \leq y \leq 250$$

METHOD OF ABSOLUTE VALUE

- The derivative is

$$A'(y) = 500 - 4y$$
$$A'(y) = 0$$
$$y = 125$$

SOLUTION 1

- The maximum area is

$$A(0) = 0$$
$$A(125) = 62500, \quad \text{maximum}$$
$$A(250) = 0$$

- Therefore the dimensions of the field that give maximum area is

$$width, y = 125\ ft$$
$$length, x = 250\ ft$$

SOLUTION 1

- Solve using the second derivative test
- The second derivative is

SECOND DERIVATIVE TEST

$$\frac{dA}{dy} = 500 - 4y$$

$$\frac{dA}{dy} = 0$$

$$y = 125$$

$$\frac{d^2A}{dy^2} = -4 < 0 \Rightarrow \max$$

$$x = 250$$

- Therefore the dimensions of the field that give maximum area is

$$width, y = 125\ ft$$

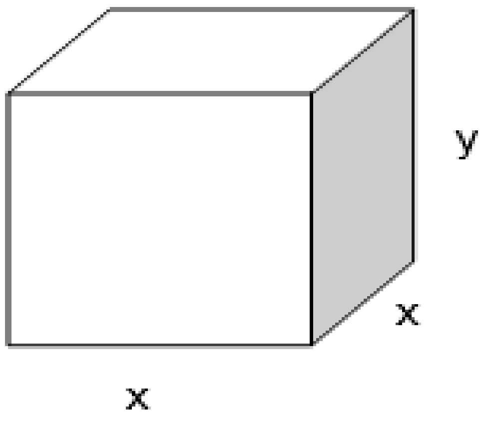
$$length, x = 250\ ft$$

EXAMPLE 2

We want to construct a box with a square base and we only have 10 m^2 of material to use in construction of the box. Assuming that all material is used in the construction process, determine the maximum volume that the box can have.

SOLUTION 2

- To find the maximum volume of the box.
- Let
  - x = length of the box with square base
  - y = height of the box



SOLUTION 2

- Express the equation of volume since we want to find the maximum volume.
- Volume = area of the base x height

$$V = x^2 y$$

$$A = 10$$

$$10 = 2x^2 + 4xy$$

$$y = \frac{5}{2x} - \frac{x}{2}$$

$$V = x^2\left(\frac{5}{2x} - \frac{x}{2}\right)$$

SOLUTION 2

- Solve using the method of absolute value
- Note that $x > 0$, so the function we wish to maximize is

$$V = x^2\left(\frac{5}{2x} - \frac{x}{2}\right), \quad x > 0$$

METHOD OF ABSOLUTE VALUE

- The derivative is

$$\frac{dV}{dx} = \frac{5}{2} - \frac{3x^2}{2}$$

$$\frac{dV}{dx} = 0$$

$$\frac{3x^2}{2} = \frac{5}{2}$$

$$x^2 = \frac{5}{3}$$

$$x = 1.2910$$

SOLUTION 2

- Therefore the maximum volume is

$$V = x^2 y$$

$$V = (1.2910)^2 (1.2910)$$

$$V = 2.1517 \text{ m}^3$$

SOLUTION 2

- Solve using the second derivative test
- The second derivative is

SECOND DERIVATIVE TEST

$$\frac{dV}{dx} = \frac{5}{2} - \frac{3x^2}{2}$$

$$\frac{d^2V}{dx^2} = -3x$$

$$= -3(1.2910)$$

$$= -3.873 < 0 \Rightarrow \max$$

$$x = 1.2910$$

$$y = 1.2910$$

- Therefore the maximum volume is

$$V = x^2 y$$

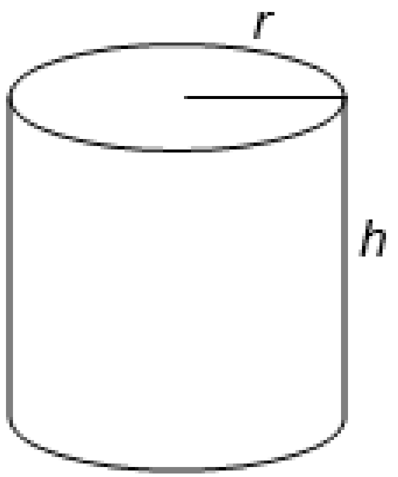
$$V = 2.1517 \text{ m}^3$$

EXAMPLE 3

A manufacturing needs to make an open top cylindrical can that will hold 1.5 litre of liquid. Determine the dimensions of the can that will minimize the amount of material used in its construction.

SOLUTION 3

- To find the dimensions (radius and height) of the can that give minimum area.
- Let
  - r = radius of the can
  - h = height of the can



SOLUTION 3

- Express the equation of area since we want to find the minimum area (amount of material used related with area).
- Area = area of the side + area of the base (open top)

$$A = 2\pi r h + \pi r^2$$

$$V = 1500\, ml$$

$$1500 = \pi r^2 h$$

$$h = \frac{1500}{\pi r^2}$$

$$A = 2\pi r\left(\frac{1500}{\pi r^2}\right) + \pi r^2$$

SOLUTION 3

- Solve using the method of absolute value
- Note that $r > 0$, so the function we wish to minimize is

$$A = \frac{3000}{r} + \pi r^2, \quad r > 0$$

METHOD OF ABSOLUTE VALUE

- The derivative is

$$\frac{dA}{dr} = -\frac{3000}{r^2} + 2\pi r$$

$$\frac{dA}{dr} = 0$$

$$2\pi r = \frac{3000}{r^2}$$

$$r^3 = \frac{3000}{2\pi}$$

$$r = 7.8159$$

SOLUTION 3

- The minimum area is

$$A = \frac{3000}{r} + \pi r^2$$

$$A = \frac{3000}{7.8159} + \pi (7.8159)^2$$

$$A = 575.7475$$

- Therefore the dimensions of the can that give minimum area is

$$radius, r = 7.8159$$

$$height, h = 7.8160$$

SOLUTION 3

- Solve using the second derivative test
- The second derivative is

SECOND DERIVATIVE TEST

$$\frac{dA}{dr} = -\frac{3000}{r^2} + 2\pi r$$

$$\frac{dA}{dr} = 0$$

$$r = 7.8159$$

$$\frac{d^2A}{dr^2} = \frac{6000}{r^3} + 2\pi$$

$$= \frac{6000}{(7.8159)^3} + 2\pi$$

$$= 18.8497 > 0 \Rightarrow \min$$

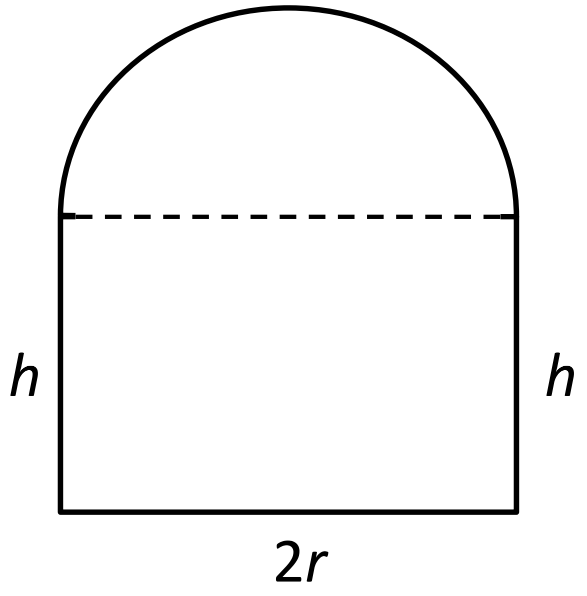
- Therefore the dimensions of the can that give minimum area is

$$radius, r = 7.8159$$

$$height, h = 7.8160$$

EXAMPLE 4

A window frame consists of a rectangle of height h metres surmounted by a semicircle of radius r metres as shown in the diagram below. If the perimeter of the frame is constant at 10 metres, find the value of r for which the area of the frame is a maximum.



SOLUTION 4

- Express the equation of area since we want to find the maximum area
- Area = area of the rectangle + area of the semicircle

$$\text{Perimeter} = 10$$

$$2h + 2r + \pi r = 10$$

$$2h = 10 - r(\pi + 2)$$

$$A = 2hr + \frac{\pi r^2}{2}$$

$$A = r[10 - r(\pi + 2)] + \frac{\pi r^2}{2}$$

$$A = 10r - 2r^2 - \frac{1}{2}\pi r^2$$

SOLUTION 4

- Solve using the method of absolute value
- Note that $r > 0$, so the function we wish to minimize is

$$A = 10r - 2r^2 - \frac{1}{2}\pi r^2, \quad r > 0$$

METHOD OF ABSOLUTE VALUE

- The derivative is

$$A = 10r - 2r^2 - \frac{1}{2}\pi r^2$$

$$\frac{dA}{dr} = 10 - 4r - \pi r$$

$$\frac{dA}{dr} = 0$$

$$r = \frac{10}{\pi + 4} = 1.4002$$

SOLUTION 4

- Solve using the second derivative test
- The second derivative is

SECOND DERIVATIVE TEST

$$\frac{dA}{dr} = 10 - 4r - \pi r$$

$$\frac{d^2A}{dr^2} = -4 - \pi < 0 \Rightarrow \max$$

- Therefore the radius that give maximum area is

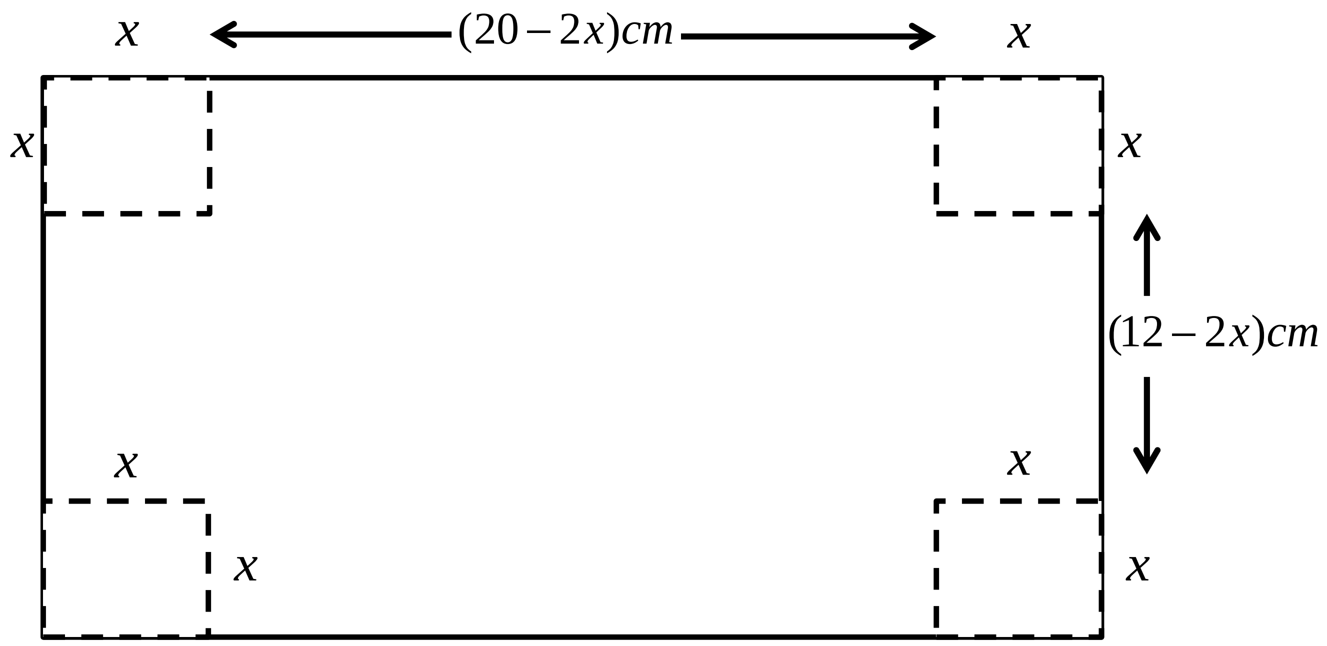
$$radius, r = 1.4002$$

EXAMPLE 5

A sheet of cardboard measures 20 cm by 12 cm. Four little squares of side lengths x *cm* are cut out from the corners of the cardboard and the remainder is turned up to form an open rectangular box. Find the value of x for which the volume of the box is a maximum.

SOLUTION 5

- To find the value of x that give maximum volume.



SOLUTION 5

- Express the equation of volume since we want to find the maximum volume
- Volume = area of the base x height

$$
\begin{aligned}
V &= (20-2x)(12-2x)x \\
&= (240-40x-24x+4x^2)x \\
&= 240x-64x^2+4x^3
\end{aligned}
$$

SOLUTION 5

- Solve using the method of absolute value
- Note that $x > 0$, so the function we wish to minimize is

$$V = (20-2x)(12-2x)x, \quad x > 0$$

METHOD OF ABSOLUTE VALUE

- The derivative is

$$\frac{dV}{dx} = 240 - 128x + 12x^2$$

$$\frac{dV}{dx} = 0$$

$$x = 22.77, 2.43$$

- For practical measures $x = 2.43$ because the longest side value is 20 cm.

SOLUTION 5

- Solve using the second derivative test
- The second derivative is

SECOND DERIVATIVE TEST

$$\frac{dV}{dx} = 240 - 128x + 12x^2$$

$$\frac{d^2V}{dx^2} = -128 + 24x$$

For $x = 2.43$

$$\frac{d^2V}{dx^2} = -69.68 < 0 \Rightarrow \max$$

For $x = 22.77$

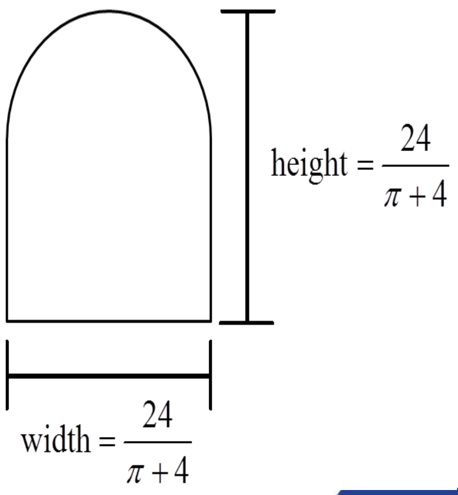
$$\frac{d^2V}{dx^2} = 418.48 > 0$$

- Therefore the value of $x = 2.43$ give maximum volume

TRY IT YOURSELF 1

A window is being built and the bottom is a rectangle and the top is a semicircle. If the window has a perimeter 12 meters, what must the dimensions of window so that the area is maximize?

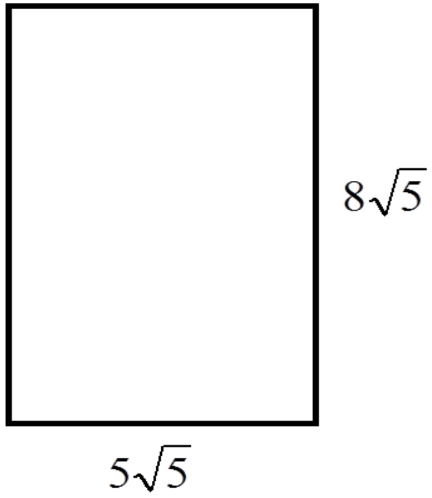
Solution



TRY IT YOURSELF 2

A printer need to make a poster that will have a total area of 200 cm^2 and will have 1 cm margins on the sides, 2 cm margin on the top and 1.5 cm margin on the bottom. What dimensions will give the largest printed area?

Solution



SUMMARY

- Understand the definition of optimization
- Distinguish between maximum or minimum problem
- Solve application problems using absolute value method
- Solve application problems using second derivative test

REFERENCES

- James, S. (2012). *Calculus* (7th ed.). Cengage Learning.
- Bivens, I.C., Stephen, D., & Howard, A. (2012). Calculus Early Transcedentals (10th ed.). John Willey & Sons Inc.