ENGINEERING MATHEMATICS 1
BMFG 1313
INTERPOLATION

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Lesson Outcome

Upon completion of this lesson, the student should be able to:

1. Determine polynomials using Newton's interpolation and Lagrange interpolation.
2. Estimate the value of a function for any intermediate value of the independent variable.

Interpolation
A method of estimating the value of a function for any intermediate value of the independent variable

Newton's Interpolating Polynomials

Lagrange Interpolating Polynomials

Curve Fitting
- Find the **polynomial function** that fits a given set of points

Use the polynomial function to constructing a new data point

5.1 Newton’s Interpolating Polynomials

- Applicable for **equally and unequally spaced** data.

- A straight line from Newton’s Interpolating Polynomial that passing through two points (x_0, y_0) and (x_1, y_1)

$$f(x) = b_0 + b_1(x - x_0)$$

5.1 Newton's Interpolating Polynomials

- A parabola from Newton's Interpolating Polynomial that passing through three points $(x_0, y_0), (x_1, y_1)$ and (x_2, y_2)

$$f(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

- **General formula** of Newton's Interpolating Polynomial that passing through n points $(x_0, y_0), (x_1, y_1), \ldots, (x_{n-1}, y_{n-1})$

$$\boxed{\begin{aligned} f(x) &= b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + \cdots \\ &\quad + b_{n-1}(x - x_0) \ldots (x - x_{n-2}) \end{aligned}}$$

5.1 Newton's Interpolating Polynomials

Example:

Construct the parabola that passing through (x_0, y_0), (x_1, y_1) and (x_2, y_2).

From Newton's Interpolation, the parabolic equation is

$$f(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

Derivation of coefficient b_0, b_1, b_2 :

Substitute (x_0, y_0) into x and y:

$$y_0 = b_0 + b_1(\underbrace{x_0 - x_0}_{0}) + b_2(\underbrace{x_0 - x_0}_{0})(x_0 - x_1)$$

$$\therefore b_0 = y_0$$

5.1 Newton's Interpolating Polynomials

Substitute (x_1, y_1) into x and y:

$$y_1 = y_0 + b_1(x_1 - x_0) + b_2(x_1 - x_0)\underbrace{(x_1 - x_1)}_{0}$$

$$\therefore b_1 = \frac{y_1 - y_0}{x_1 - x_0}$$

Substitute (x_2, y_2) into x and y:

$$y_2 = y_0 + \frac{y_1 - y_0}{x_1 - x_0}(x_2 - x_0) + b_2(x_2 - x_0)(x_2 - x_1)$$

$$\therefore b_2 = \frac{y_2 - y_0 - \frac{y_1 - y_0}{x_1 - x_0}(x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)} = \frac{\frac{y_2 - y_1}{x_2 - x_1} - \frac{y_1 - y_0}{x_1 - x_0}}{(x_2 - x_0)}$$

5.1 Newton's Interpolating Polynomials

i	x_i	$f(x_i)$	Δf_i	$\Delta^2 f_i$	$\Delta^3 f_i$	$\Delta^4 f_i$
		b_0	b_1	b_2	b_3	b_4
0	x_0	$\boxed{y_0}$				
			$\boxed{\Delta f_0 = \frac{y_1 - y_0}{x_1 - x_0}}$			
1	x_1	y_1		$\boxed{\Delta^2 f_0 = \frac{\Delta f_1 - \Delta f_0}{x_2 - x_0}}$		
			$\Delta f_1 = \frac{y_2 - y_1}{x_2 - x_1}$		$\boxed{\Delta^3 f_0 = \frac{\Delta^2 f_1 - \Delta^2 f_0}{x_3 - x_0}}$	
2	x_2	y_2		$\Delta^2 f_1 = \frac{\Delta f_2 - \Delta f_1}{x_3 - x_1}$		$\boxed{\Delta^4 f_0 = \frac{\Delta^3 f_1 - \Delta^3 f_0}{x_4 - x_0}}$
			$\Delta f_2 = \frac{y_3 - y_2}{x_3 - x_2}$		$\Delta^3 f_1 = \frac{\Delta^2 f_2 - \Delta^2 f_1}{x_4 - x_1}$	
3	x_3	y_3		$\Delta^2 f_2 = \frac{\Delta f_3 - \Delta f_2}{x_4 - x_2}$		
			$\Delta f_3 = \frac{y_4 - y_3}{x_4 - x_3}$			
4	x_4	y_4				

5.1 Newton's Interpolating Polynomials

5.1.1 Linear Interpolation

To connect **two data points** with the first order polynomial (straight line)

Formula:

$$f(x) = b_0 + b_1(x - x_0)$$

Table:

i	x_i	$f(x_i)$	Δf_i
0	x_0	$\boxed{y_0}$	
			$\boxed{\Delta f_0 = \dfrac{y_1 - y_0}{x_1 - x_0}}$
1	x_1	y_1	

5.1 Newton's Interpolating Polynomials
5.1.1 Linear Interpolation

Example:

Estimate natural logarithm of 2 ($\ln 2$) using linear interpolation given $\ln 1 = 0$ and $\ln 3 = 1.0986$.

Solution: $(1, 0), (3, 1.0986), (2, ?)$

i	x_i	$f(x_i)$	Δf_i
0	1	$\boxed{0.0000}$ → b_0	
			$\Delta f_0 = \dfrac{1.0986 - 0}{3 - 1} = \boxed{0.5493}$ → b_1
1	3	1.0986	

$$f(x) = b_0 + b_1(x - x_0) = 0.5493(x - 1) \quad \approx \ln x$$

$$\therefore f(2) = 0.5493$$

5.1 Newton's Interpolating Polynomials
5.1.2 Quadratic Interpolation

To connect **three data points** with the second order polynomial (parabola)

Formula:

$$f(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

Table:

i	x_i	$f(x_i)$	Δf_i	$\Delta^2 f_i$
0	x_0	$\boxed{y_0}$		
			$\boxed{\Delta f_0 = \frac{y_1 - y_0}{x_1 - x_0}}$	
1	x_1	y_1		$\boxed{\Delta^2 f_0 = \frac{\Delta f_1 - \Delta f_0}{x_2 - x_0}}$
			$\Delta f_1 = \frac{y_2 - y_1}{x_2 - x_1}$	
2	x_2	y_2		

5.1.2 Quadratic Interpolation

Example:

Estimate $f(1.7)$ using quadratic interpolation given $(0, 1), (2, -3), (4, -8)$.

i	x_i	$f(x_i)$	Δf_i	$\Delta^2 f_i$	
0	0	$\boxed{1}$			$\rightarrow b_0$
			$\frac{-3-1}{2-0} = \boxed{-2}$		$\rightarrow b_1$
1	2	-3		$\frac{-2.5-(-2)}{4-0} = \boxed{-0.125}$	$\rightarrow b_2$
			$\frac{-8-(-3)}{4-2} = -2.5$		
2	4	-8			

$$\begin{aligned} f(x) &= b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) \\ &= 1 - 2x - 0.125x(x - 2) \end{aligned}$$

$$\therefore f(1.7) = -2.3363$$

5.1 Newton's Interpolating Polynomials
5.1.3 Cubic Interpolation

To connect **four data points** with the third order polynomial (cubic polynomial)

Formula:

$$f(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + b_3(x - x_0)(x - x_1)(x - x_2)$$

Table:

i	x_i	$f(x_i)$	Δf_i	$\Delta^2 f_i$	$\Delta^3 f_i$
0	x_0	$\boxed{y_0}$			
			$\boxed{\Delta f_0 = \frac{y_1 - y_0}{x_1 - x_0}}$		
1	x_1	y_1		$\boxed{\Delta^2 f_0 = \frac{\Delta f_1 - \Delta f_0}{x_2 - x_0}}$	
			$\Delta f_1 = \frac{y_2 - y_1}{x_2 - x_1}$		$\boxed{\Delta^3 f_0 = \frac{\Delta^2 f_1 - \Delta^2 f_0}{x_3 - x_0}}$
2	x_2	y_2		$\Delta^2 f_1 = \frac{\Delta f_2 - \Delta f_1}{x_3 - x_1}$	
			$\Delta f_2 = \frac{y_3 - y_2}{x_3 - x_2}$		
3	x_3	y_3			

5.1.3 Cubic Interpolation

Example:

Estimate $f(5.3)$ using cubic interpolation given $(1,1), (4,9), (7,13), (10,20)$.

i	x_i	$f(x_i)$	Δf_i	$\Delta^2 f_i$	$\Delta^3 f_i$	
0	1	$\boxed{1}$				$\rightarrow b_0$
			$\boxed{2.6667}$			$\rightarrow b_1$
1	4	9		$\boxed{-0.2222}$		$\rightarrow b_2$
			1.3333		$\boxed{0.0432}$	$\rightarrow b_3$
2	7	13		0.1667		
			2.3333			
3	10	20				

$$f(x) = 1 + 2.6667(x-1) - 0.2222(x-1)(x-4) + 0.0432(x-1)(x-4)(x-7)$$

$$\therefore f(5.3) = 10.8142$$

5.1.3 Cubic Interpolation

Exercise 5.1:

Given the following data

i	0	1	2	3
x_i	0	1	2	3
$f(x_i)$	10	8	2	-9

1) By using Newton's Interpolating polynomial of order 2, estimate $f(0.6)$.
2) By using Newton's Interpolation, find the $P_3(x)$ and then use it to estimate $f(2.7)$.

[Ans: 9.28; -5.1156]

5.2 Lagrange Interpolating Polynomials

- Applicable for **equally spaced or not equally spaced** data.

- A straight line from Lagrange Interpolating Polynomial that passing through two points (x_0, y_0) and (x_1, y_1)

$$f(x) = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1)$$

5.2 Lagrange Interpolating Polynomials

- A parabola from Lagrange Interpolating Polynomial that passing through three points (x_0, y_0), (x_1, y_1) and (x_2, y_2)

$$f(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2)$$

5.2 Lagrange Interpolating Polynomials

General formula of Lagrange Interpolating Polynomial that passing through n points $(x_0, y_0), (x_1, y_1), \ldots, (x_{n-1}, y_{n-1})$:

n is order of interpolation

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i) = L_0(x) f(x_0) + \cdots + L_n(x) f(x_n)$$

where

$$L_i(x) = \prod_{j=0, j\neq i}^{n} \frac{(x - x_j)}{(x_i - x_j)}$$

$$= \frac{(x - x_0)(x - x_1)}{(x_i - x_0)(x_i - x_1)} \cdots \frac{(x - x_{i-1})(x - x_{i+1})}{(x_i - x_{i-1})(x_i - x_{i+1})} \cdots \frac{(x - x_n)}{(x_i - x_n)}$$

5.2 Lagrange Interpolating Polynomials

Lagrange Interpolation Coefficient:

$$L_i(x) = \prod_{j=0, j\neq i}^{n} \frac{(x - x_j)}{(x_i - x_j)}$$

is a polynomial with degree *n* which satisfies

$$L_i(x_j) = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

and

$$\sum_{i=0}^{n} L_i(x_j) = 1$$

5.2 Lagrange Interpolating Polynomials

5.2.1 Linear Interpolation

To connect **two data points** with the first order polynomial (straight line)

Formula:

$$f(x) = L_0(x)f(x_0) + L_1(x)f(x_1)$$

where

$$L_0(x) = \frac{x - x_1}{x_0 - x_1}$$

$$L_1(x) = \frac{x - x_0}{x_1 - x_0}$$

5.2.1 Linear Interpolation

Example:

Use Lagrange Interpolating Polynomial of degree 1 to estimate $f(4.7)$ given $f(4.3) = 0.1678$ and $f(4.9) = 0.2451$.

Solution:

$$f(x) = L_0(x)f(x_0) + L_1(x)f(x_1)$$

where

$$L_0(x) = \frac{x - x_1}{x_0 - x_1} = \frac{x - 4.9}{4.3 - 4.9}$$

$$L_1(x) = \frac{x - x_0}{x_1 - x_0} = \frac{x - 4.3}{4.9 - 4.3}$$

5.2.1 Linear Interpolation

Solution: (cont.)

Recall:

$f(4.3) = 0.1678$

$f(4.9) = 0.2451$

$$f(x) = L_0(x)f(x_0) + L_1(x)f(x_1)$$

$$f(x) = \frac{x - 4.9}{4.3 - 4.9}(0.1678) + \frac{x - 4.3}{4.9 - 4.3}(0.2451)$$

$$f(4.7) = \frac{4.7 - 4.9}{4.3 - 4.9}(0.1678) + \frac{4.7 - 4.3}{4.9 - 4.3}(0.2451) = 0.2193$$

5.2 Lagrange Interpolating Polynomials
5.2.2 Quadratic Interpolation

To connect **three data points** with the second order polynomial (parabola)

Formula:

$$f(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2)$$

where

$$L_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)}$$

$$L_1(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)}$$

$$L_2(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}$$

5.2.2 Quadratic Interpolation

Example:

Use Lagrange Interpolating Polynomial of degree 2 to estimate $f(-3.8)$ given $f(-10.3) = 1.0823$, $f(-5.1) = 4.7798$ and $f(-1.5) = 9.1514$.

Solution:

$$f(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2)$$

where

$$L_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} = \frac{(x + 5.1)(x + 1.5)}{(-10.3 + 5.1)(-10.3 + 1.5)}$$

$$L_1(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} = \frac{(x + 10.3)(x + 1.5)}{(-5.1 + 10.3)(-5.1 + 1.5)}$$

$$L_2(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} = \frac{(x + 10.3)(x + 5.1)}{(-1.5 + 10.3)(-1.5 + 5.1)}$$

5.2.2 Quadratic Interpolation

Solution: (cont.)

Recall:

$$f(-10.3) = 1.0823$$
$$f(-5.1) = 4.7798$$
$$f(-1.5) = 9.1514$$

$$f(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2)$$

$$L_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} = \frac{(x + 5.1)(x + 1.5)}{(-10.3 + 5.1)(-10.3 + 1.5)} = \frac{(x + 5.1)(x + 1.5)}{45.76}$$

$$L_1(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} = \frac{(x + 10.3)(x + 1.5)}{(-5.1 + 10.3)(-5.1 + 1.5)} = -\frac{(x + 10.3)(x + 1.5)}{18.72}$$

$$L_2(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} = \frac{(x + 10.3)(x + 5.1)}{(-1.5 + 10.3)(-1.5 + 5.1)} = \frac{(x + 10.3)(x + 5.1)}{31.68}$$

5.2.2 Quadratic Interpolation

Solution: (cont.)

$$f(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2)$$

$$= \frac{(x+5.1)(x+1.5)}{45.76}(1.0823) - \frac{(x+10.3)(x+1.5)}{18.72}(4.7798)$$

$$+ \frac{(x+10.3)(x+5.1)}{31.68}(9.1514)$$

$$\therefore f(-3.8) = \frac{(-3.8+5.1)(-3.8+1.5)}{45.76}(1.0823) - \frac{(-3.8+10.3)(-3.8+1.5)}{18.72}(4.7798)$$

$$+ \frac{(-3.8+10.3)(-3.8+5.1)}{31.68}(9.1514)$$

$$= 6.1874$$

5.2 Lagrange Interpolating Polynomials
5.2.2 Quadratic Interpolation

Exercise 5.2:

Given the following data

i	0	1	2
x_i	-1	1	3
$f(x_i)$	11	8	3

By using Lagrange Interpolating polynomial of order 2, estimate $f(1.8)$.

[Ans: 6.24]

5.2 Lagrange Interpolating Polynomials
5.2.3 Cubic Interpolation

To connect **four data points** with the third order polynomial (cubic polynomial)

Formula:

$$f(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2) + L_3(x)f(x_3)$$

where

$$L_0(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}$$

$$L_3(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}$$

5.2 Lagrange Interpolating Polynomials
5.2.3 Cubic Interpolation

Example:

Construct the Lagrange Interpolation polynomial of degree 3 with the data set given below:

i	0	1	2	3
x_i	-1	0	1	2
$f(x_i)$	1	1	1	-5

5.2 Lagrange Interpolating Polynomials

5.2.3 Cubic Interpolation

Solution:

$$f(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2) + L_3(x)f(x_3)$$

$$= \frac{(x-0)(x-1)(x-2)}{(-1-0)(-1-1)(-1-2)}(1) + \frac{(x+1)(x-1)(x-2)}{(0+1)(0-1)(0-2)}(1)$$

$$+ \frac{(x+1)(x-0)(x-2)}{(1+1)(1-0)(1-2)}(1) + \frac{(x+1)(x-0)(x-1)}{(2+1)(2-0)(2-1)}(-5)$$

$$= -\frac{x}{6}(x-1)(x-2) + \frac{1}{2}(x+1)(x-1)(x-2)$$

$$-\frac{x}{2}(x+1)(x-2) - \frac{5x}{6}(x+1)(x-1)$$

$$\therefore f(x) = 1 + x - x^3$$

5.2 Lagrange Interpolating Polynomials
5.2.3 Cubic Interpolation

Exercise 5.3:

Find the Lagrange approximation polynomial of degree 3 for the function

$$f(x) = \frac{x^2 - e^x + \ln x}{\sqrt{x + \sin x}}.$$

Use the $x_0 = 1, x_1 = 3, x_2 = 4$ and $x_3 = 7$. Then estimate $f(5)$.

[Ans: -73.2355]