ENGINEERING MATHEMATICS 1
BMFG 1313
COMPLEX NUMBER
- Introduction and Its Properties
- Powers of the Complex Number

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Learning Outcomes

Upon completion of this lesson, the student should be able to:

1. Use the properties of complex number to solve an equation.
2. Apply Euler's and De Moivre's method to solve operations of complex number.

4.1 Introduction and its Properties

Complex numbers have practical applications in many fields, including biology, chemistry, physics, economics, electrical engineering, and statistics.

A **complex number** is expressed in the **Cartesian form**

$$z = a + \mathrm{j}b$$

where a and b are real numbers and i is the imaginary unit, where $\mathrm{j}^2 = -1$.

The number a is the **real part** of z, denoted by Re z, and b is the **imaginary part** of z, denoted by Im z.

4.1 Introduction and its Properties

For example:

a) Given $z = 5 - \text{j}6$, thus,

Re $z = 5$ and Im $z = -6$

b) Given $z = -2 + \text{j}3$, thus,

Re $z = -2$ and Im $z = 3$

The set of all complex numbers is denoted by $\mathbb{C}$.

4.1 Introduction and its Properties

Complex Roots:

A quadratic equation $ax^2 + bx + c = 0$ has complex roots when $b^2 - 4ac < 0$.

Example:

Find the root of $x^2 - 4x + 5 = 0$.

Solution:

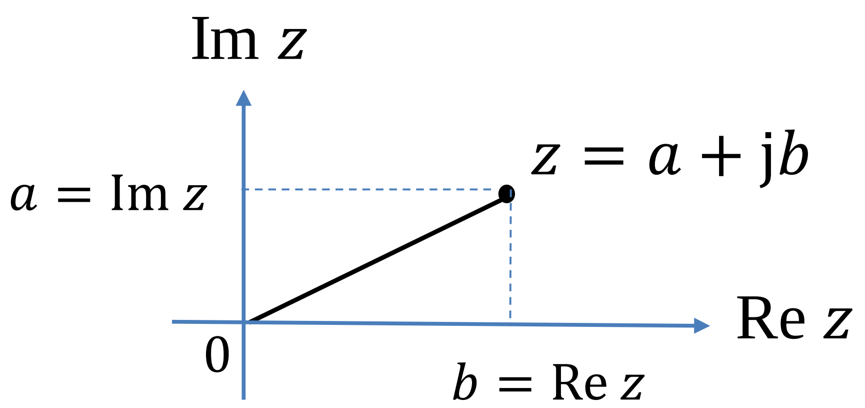
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{4 \pm \sqrt{16 - 4(1)(5)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{-4}}{2} = \frac{4 \pm j2}{2} = 2 \pm j$$

$$\sqrt{-4} = \sqrt{-1 \times 4} = \sqrt{-1} \times \sqrt{4} = j2$$

4.1.1 Argand Diagram

Complex numbers can be represented geometrically similar to a point in Cartesian Coordinates system as follows:

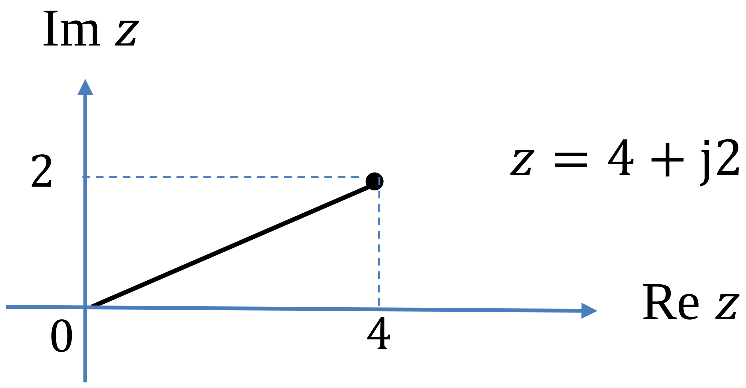


Argand Diagram of $z = a + \mathrm{j}b$

4.1.1 Argand Diagram

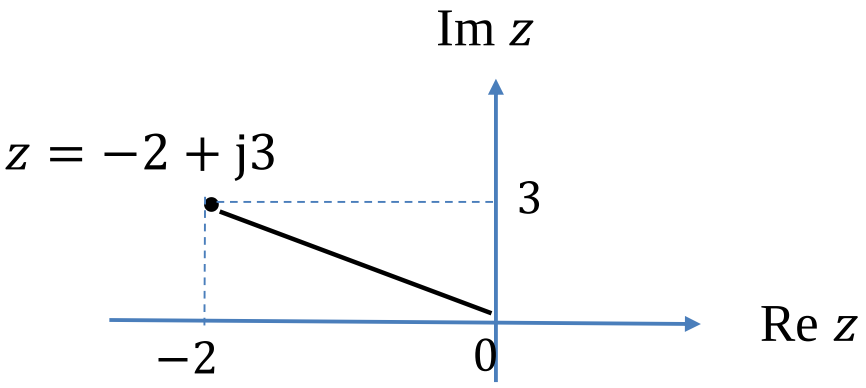
Example:

Argand diagram of $z = 4 + j2$ is



Example:

Argand diagram of $z = -2 + j3$ is



4.1.2 Equality

If $z_1 = a_1 + \mathrm{j}b_1$ is equal to $z_2 = a_2 + \mathrm{j}b_2$, then $a_1 = a_2$ and $b_1 = b_2$.

Example:

Find the values of p and q if $z_1 = (2 - p) - \mathrm{j}5$ and $z_2 = 10 + \mathrm{j}(q + 4)$ are equal.

Solution:

$$2 - p = 10 \quad \Rightarrow \quad p = -8$$
$$q + 4 = -5 \quad \Rightarrow \quad q = -9$$

4.1.3 Addition and Subtraction

If $z_1 = a_1 + \mathrm{j}b_1$ and $z_2 = a_2 + \mathrm{j}b_2$, then

$$z_1 + z_2 = (a_1 + a_2) + \mathrm{j}(b_1 + b_2)$$

and

$$z_1 - z_2 = (a_1 - a_2) + \mathrm{j}(b_1 - b_2)$$

4.1.3 Addition and Subtraction

Example:

Given $z_1 = 13 - \text{j}6$ and $z_2 = 7 + \text{j}12$, find $z_1 + z_2$ and $z_1 - z_2$.

Solution:

$$z_1 + z_2 = (13 + 7) + \text{j}(-6 + 12) = 20 + \text{j}6$$

$$z_1 - z_2 = (13 - 7) + \text{j}(-6 - 12) = 6 - \text{j}18$$

4.1.3 Addition and Subtraction

Example:

Given $z = z_1 + z_2 = 2 - j4$. If $z_1 = -1 + j3$, find z_2.

Solution:

$$-1 + j3 + z_2 = 2 - j4$$

$$z_2 = 2 - j4 - (-1 + j3)$$

$$= 3 - j7$$

4.1.4 Multiplication

If $z_1 = a_1 + \mathrm{j}b_1$ and $z_2 = a_2 + \mathrm{j}b_2$, then

$$\begin{aligned} z_1 z_2 &= (a_1 + \mathrm{j}b_1)(a_2 + \mathrm{j}b_2) \\ &= a_1 a_2 + \mathrm{j}a_1 b_2 + \mathrm{j}a_2 b_1 + \mathrm{j}^2 b_1 b_2 \\ &= (a_1 a_2 - b_1 b_2) + \mathrm{j}(a_1 b_2 + a_2 b_1) \end{aligned}$$

$\mathrm{j}^2 = -1$

4.1.4 Multiplication

Example:

Given $z_1 = -2 + \mathrm{j}3$ and $z_2 = 4 + \mathrm{j}7$, find $z_1 z_2$.

Solution:

$$z_1 z_2 = (-2 + \mathrm{j}3)(4 + \mathrm{j}7)$$

$$= -8 - \mathrm{j}14 + \mathrm{j}12 + \mathrm{j}^2 21$$

$$\mathrm{j}^2 = -1$$

$$= -29 - \mathrm{j}2$$

4.1.4 Multiplication

Example:

Find $z_1 z_2$ if $z_1 = 1 - j2$ and $z_2 = -2 - j2$.

Solution:

$$z_1 z_2 = (1 - j2)(-2 - j2)$$

$$= -2 - j2 + j4 + j^2 4$$

$j^2 = -1$

$$= -6 + j2$$

4.1.5 Division

If $z_1 = a_1 + \mathrm{j}b_1$ and $z_2 = a_2 + \mathrm{j}b_2$, then

Multiply "top and bottom" with conjugate of denominator

$$\frac{z_1}{z_2} = \frac{(a_1 + \mathrm{j}b_1)}{(a_2 + \mathrm{j}b_2)} \times \boxed{\frac{(a_2 - \mathrm{j}b_2)}{(a_2 - \mathrm{j}b_2)}}$$

$$= \frac{(a_1a_2 + b_1b_2) + \mathrm{j}(-a_1b_2 + b_1a_2)}{a_2^2 + b_2^2}$$

4.1.5 Division

Example:

Given $z_1 = -2 + \mathrm{j}3$ and $z_2 = 4 + \mathrm{j}7$, find $\frac{z_1}{z_2}$.

Solution:

$$\frac{z_1}{z_2} = \frac{(-2+\mathrm{j}3)}{(4+\mathrm{j}7)} \times \frac{(4-\mathrm{j}7)}{(4-\mathrm{j}7)} = \frac{-8+\mathrm{j}14+\mathrm{j}12-\mathrm{j}^2 21}{16+49}$$

$$= \frac{13+\mathrm{j}26}{65} = 5 + \mathrm{j}\frac{5}{2}$$

Multiply "top and bottom" with conjugate of denominator

4.1.5 Division

Example:

Find $z = \frac{z_1}{z_2}$ if $z_1 = 1 - \text{j}2$ and $z_2 = -2 - \text{j}2$.

Solution:

$$z = \frac{(1 - \text{j}2)}{(-2 - \text{j}2)} \times \frac{(-2 + \text{j}2)}{(-2 + \text{j}2)} = \frac{-2 + \text{j}2 + \text{j}4 - \text{j}^2 4}{4 + 4}$$

$$= \frac{2 + \text{j}6}{8} = \frac{1}{4} + \text{j}\frac{3}{4}$$

Multiply "top and bottom" with conjugate of denominator

4.1.6 Complex Conjugate

The complex conjugate of $z = a + \mathrm{j}b$ is $\bar{z} = a - \mathrm{j}b$ (by reversing the sign of the imaginary part).

In the Argand diagram, conjugate of a complex number is the image of reflection of the complex number over x-axis.

Some properties:

$$z + \bar{z} = 2a$$

$$z - \bar{z} = 2\mathrm{j}b$$

$$z\bar{z} = a^2 + b^2$$

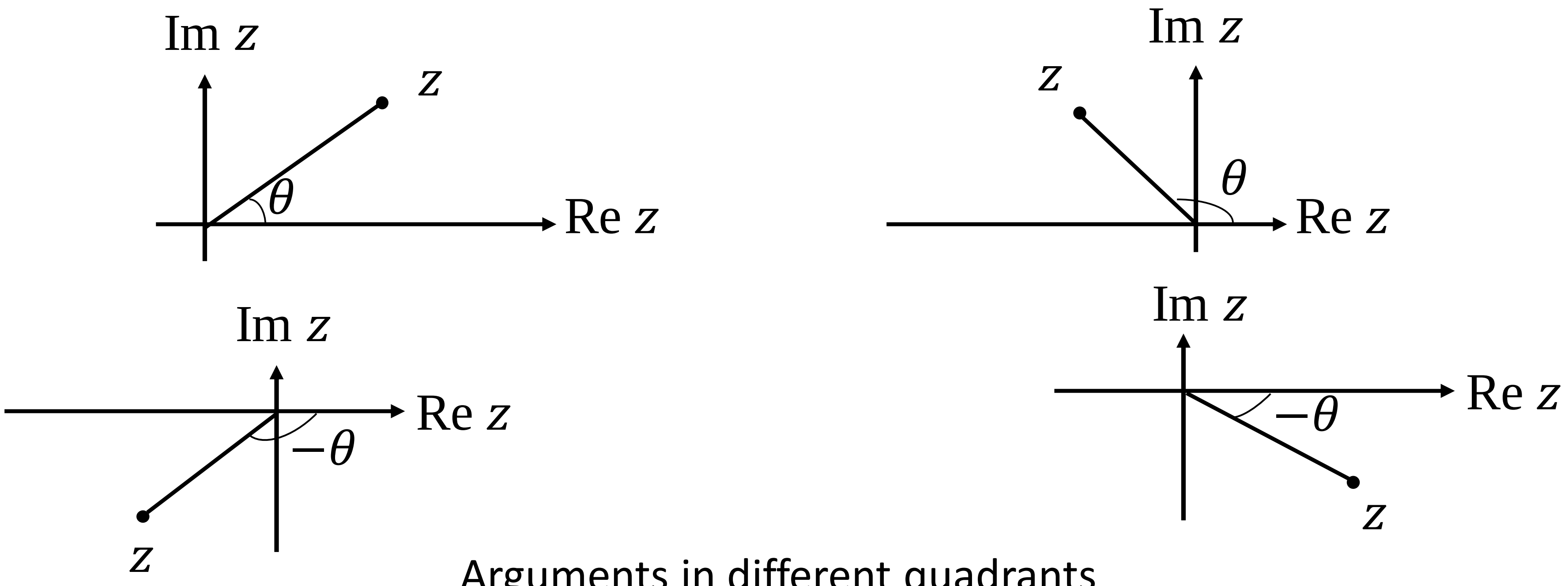
$$\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$$

4.1.7 Modulus and Argument

The **modulus** of $z = a + \text{j}b$ is defined by

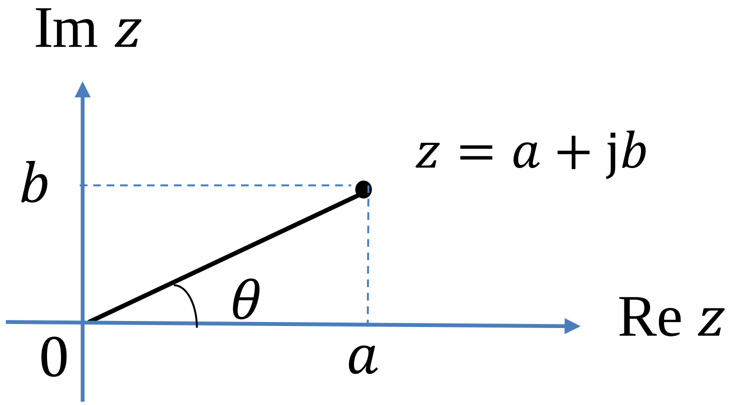
$$|z| = \sqrt{a^2 + b^2}$$

Argument of $z = a + \text{j}b$ is the **angle** between the positive real axis and the point (a, b), as shown below, where $-\pi \leq \theta \leq \pi$.



Arguments in different quadrants

4.1.7 Modulus and Argument



Argument of $z = a + \mathrm{j}b$ is written as arg $z = \theta$

and

$$\theta = \tan^{-1}\frac{b}{a}.$$

4.1.7 Modulus and Argument

Example:

Given $z = 2 + \text{j}3$, find the modulus and argument of z.

Solution:

$$|z| = \sqrt{a^2 + b^2} = \sqrt{(2)^2 + (3)^2} = \sqrt{13}$$

$$\arg z = \tan^{-1}\frac{3}{2} = 0.9828$$

4.1.7 Modulus and Argument

Example:

Find the modulus and argument of

a) $-3 + j5$

b) $-4 - j4$

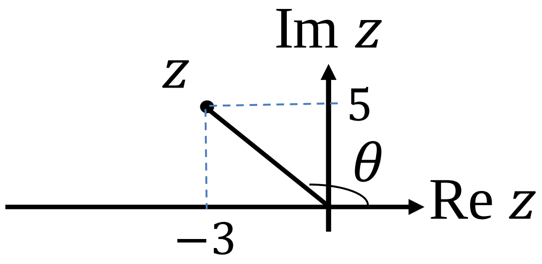
c) $8 - j5$

4.1.7 Modulus and Argument

Solution:

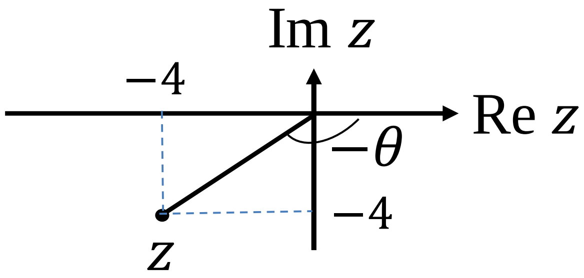
a) $|z| = \sqrt{(-3)^2 + (5)^2} = \sqrt{34}$

$$\arg z = \pi - \tan^{-1}\frac{5}{3} = 2.1112$$



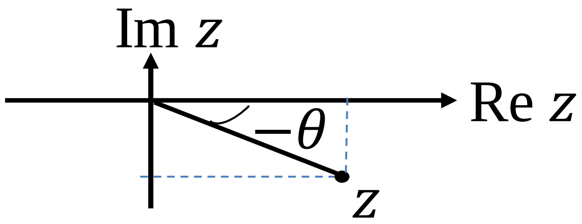
b) $|z| = \sqrt{(-4)^2 + (-4)^2} = \sqrt{32}$

$$\arg z = -\left(\pi - \tan^{-1}\frac{4}{4}\right) = -\frac{3}{4}$$



c) $|z| = \sqrt{(8)^2 + (-5)^2} = \sqrt{89}$

$$\arg z = -\left(\tan^{-1}\frac{5}{8}\right) = -0.5586$$



Exercise 4.1:

1) Find the roots of $-5x^2 + x - 2 = 0$.

2) Sketch the Argand diagram for $z = -2 + \mathrm{j}6$ and $z = 4 - \mathrm{j}5$.

3) Find the values of u and v if $z_1 = (2 + u) - \mathrm{j}(v - 4)$ and $z_2 = (2v - 3) + \mathrm{j}(u + 1)$ are equal.

4) Given $z_1 = 8 - \mathrm{j}2$ and $z_2 = -1 + \mathrm{j}4$, find $z_1 + z_2$, $z_1 - z_2$, $z_1 z_2$ and $\frac{z_1}{z_2}$.

5) Express in the form $z = a + \mathrm{j}b$ for $(3 - \mathrm{j}5)(-2 + \mathrm{j}3)$ and $\frac{(-8+\mathrm{j}4)}{(1-\mathrm{j}2)}$ respectively.

[Ans: $\frac{1}{10} \pm \mathrm{j}\frac{\sqrt{39}}{10}$; $v = \frac{8}{3}, u = \frac{1}{3}$; $7 + \mathrm{j}2, 9 - \mathrm{j}6, \mathrm{j}34, -\frac{16}{17} - \mathrm{j}\frac{30}{17}$; $9 + \mathrm{j}19, -\frac{16}{5} - \mathrm{j}\frac{12}{5}$]

Exercise 4.1:

6) Express $\frac{1}{3-\mathrm{j}} - \frac{\mathrm{j}4}{-1+\mathrm{j}2}$ in the form of $z = a + \mathrm{j}b$.

7) Given $z = 3 - \mathrm{j}4$, express $z\bar{z}$ in the form of $z = a + \mathrm{j}b$.

8) Find the modulus and argument of

a) $15 + \mathrm{j}20$

b) $8 - \mathrm{j}4$

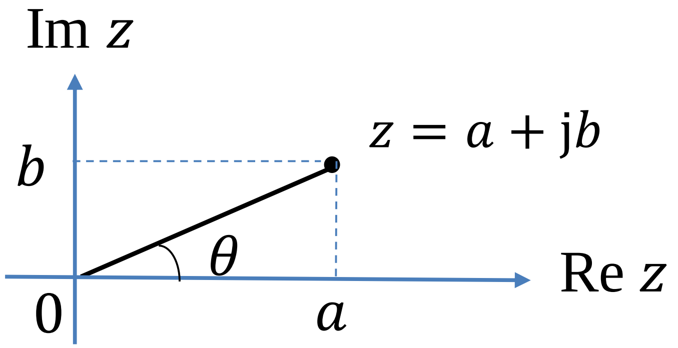
c) $-13 + \mathrm{j}9$

d) $-6 - \mathrm{j}3$

[Ans: $-\frac{13}{10} + \mathrm{j}\frac{9}{10}$; 25; $25(0.9273), \sqrt{80}(-0.4636), \sqrt{250}(2.5360), \sqrt{45}(-2.6779)$]

4.2 Powers of Complex Numbers

4.2.1 Polar Form of a Complex Number



From the diagram above, let $r = |z|$, we have

$$a = r\cos\theta \text{ and } b = r\sin\theta$$

Hence, a complex number can be expressed in **polar form** as follows:

$$z = r\cos\theta + \mathrm{j}\, r\sin\theta$$

$$\boxed{z = r(\cos\theta + \mathrm{j}\sin\theta)}$$

4.2.1 Polar Form of a Complex Number

Example:

Express the following complex numbers in polar form.

a) $-3 + j4$

b) $4 - j2$

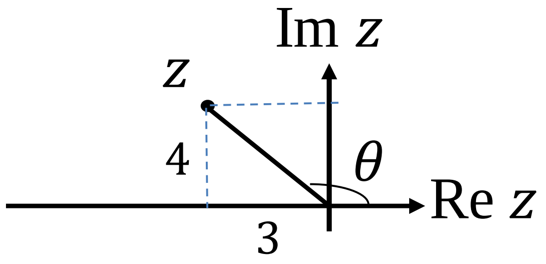
4.2.1 Polar Form of a Complex Number

Solution:

a) $|z| = \sqrt{(-3)^2+(4)^2} = 5$

$$\arg z = \pi - \tan^{-1}\frac{4}{3} = 2.2143$$

$$-3 + j4 = 5(\cos(2.2143) + j\sin(2.2143))$$

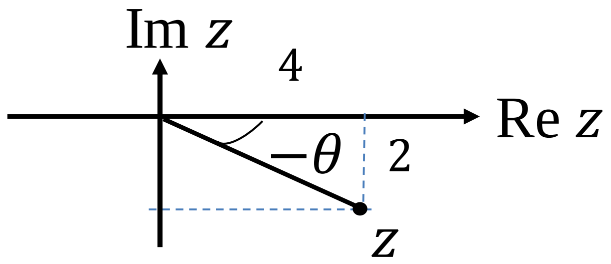


b) $|z| = \sqrt{(4)^2+(-2)^2} = \sqrt{20}$

$$\arg z = -\left(\tan^{-1}\frac{2}{4}\right) = -0.4634$$

$$4 - j2 = \sqrt{20}(\cos(-0.4634) + j\sin(-0.4634))$$

$$= \sqrt{20}(\cos(0.4634) - j\sin(0.4634))$$



4.2.2 Exponential Form of a Complex Number

This formula is known as **Euler's formula**:

$$e^{j\theta} = \cos\theta + j\sin\theta$$

By substituting Euler's formula into polar form,

$$z = r(\cos\theta + j\sin\theta)$$

$$z = re^{j\theta}$$

and this is known as **exponential form** of the complex number, z.

4.2.2 Exponential Form of a Complex Number

Example:

Express the following complex numbers in exponential form.

a) $-3 + j4$

b) $4 - j2$

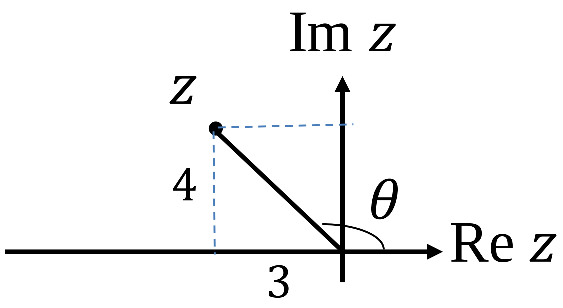
4.2.2 Exponential Form of a Complex Number

Solution:

a) $|z| = \sqrt{(-3)^2+(4)^2} = 5$

$\arg z = \pi - \tan^{-1}\frac{4}{3} = 2.2143$

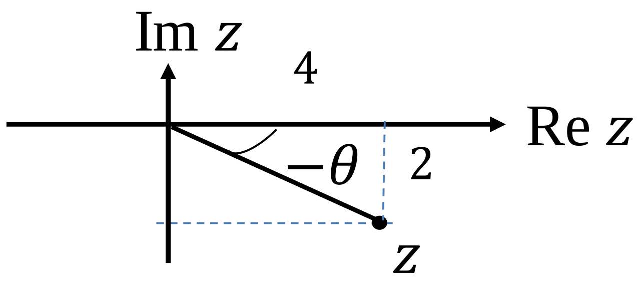
$-3 + j4 = 5e^{j2.2143}$



b) $|z| = \sqrt{(4)^2+(-2)^2} = \sqrt{20}$

$\arg z = -\left(\tan^{-1}\frac{2}{4}\right) = -0.4634$

$4 - j2 = \sqrt{20}e^{j(-0.4634)} = \sqrt{20}e^{-j0.4634}$



4.2.2 Exponential Form of a Complex Number

Example:

Express the following complex numbers in Cartesian form.

a) $12e^{j0.3578}$

b) $9e^{2-j3.1213}$

4.2.2 Exponential Form of a Complex Number

Solution:

a) $r = 12, \theta = 0.3578$

$$z = r(\cos\theta + \mathrm{j}\sin\theta)$$
$$= 12(\cos 0.3578 + \mathrm{j}\sin 0.3578)$$
$$= 11.2400 + \mathrm{j}4.2026$$

b) $r = 9e^2, \theta = -3.1213$

$$z = r(\cos\theta + \mathrm{j}\sin\theta)$$
$$= 9e^2(\cos(-3.1213) + \mathrm{j}\sin(-3.1213))$$
$$= -66.4878 - \mathrm{j}1.3494$$

$$9e^{2-\mathrm{j}3.1213} = 9e^2 e^{-\mathrm{j}3.1213}$$

4.2.3 Power of Complex Numbers

From the exponential form

$$z = re^{j\theta}$$

The power of *n* of the equation is

$$z^n = \left[re^{j\theta}\right]^n$$
$$z^n = r^n e^{jn\theta}$$

By using the Euler's formula, for any n,

$$\boxed{z^n = r^n(\cos n\theta + j\sin n\theta)}$$

and this is known as **De Moivre's Theorem**

4.2.3 Power of Complex Numbers

Example:

Express $5 - j2$ in polar form and then evaluate $(5 - j2)^4$.

Solution:

$$r = \sqrt{5^2 + (-2)^2} = \sqrt{29}, \qquad \theta = -\tan^{-1}\frac{2}{5} = -0.3805$$

$$z = r(\cos\theta + j\sin\theta) = \sqrt{29}(\cos(-0.3805) + j\sin(-0.3805))$$

By using De Moivre's theorem,

$$z^4 = r^4(\cos 4\theta + j\sin 4\theta)$$

$$(5 - j2)^4 = (\sqrt{29})^4(\cos(-1.522) + j\sin(-1.522))$$

$$= 841(0.0488 - j0.9988)$$

$$= 41.0408 - j839.9908$$

4.2.3 Power of Complex Numbers

Commonly, De Moivre's theorem is used to find the roots of complex numbers like $\sqrt{z}$ and $\sqrt[3]{z}$. In general, we want to find the nth root of z, which is $z^{1/n}$, where n is natural numbers. By setting $w = z^{1/n}$ (which gives $\mathrm{z} = w^n$) and some mathematical proving, we have

$$z^{1/n} = r^{1/n}\left[\cos\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right) + \mathrm{j}\sin\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right)\right], k = 0,1,\ldots,n-1$$

and

$$z^{1/n} = r^{1/n}e^{\mathrm{j}\left(\frac{\theta}{n}+\frac{2\pi k}{n}\right)}, k = 0,1,\ldots,n-1$$

4.2.3 Power of Complex Numbers

Example:

Given $z = -3 + \mathrm{j}2$, evaluate $z^{1/2}$.

Solution:

$r = \sqrt{(-3)^2+(2)^2} = \sqrt{13}$ and $\theta = \pi - \tan^{-1}\frac{2}{3} = 2.5536$

From formula $z^{1/n} = r^{1/n}\left[\cos\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right) + \mathrm{j}\sin\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right)\right]$,

$$z^{1/2} = (\sqrt{13})^{1/2}\left[\cos\left(\frac{2.5536}{2} + \frac{2\pi k}{2}\right) + \mathrm{j}\sin\left(\frac{2.5536}{2} + \frac{2\pi k}{2}\right)\right], k = 0,1$$

4.2.3 Power of Complex Numbers

For $k = 0,$

$$z^{1/2} = (\sqrt{13})^{1/2}[\cos(1.2768) + \mathrm{j}\sin(1.2768)] = 0.5502 + \mathrm{j}1.8174$$

For $k = 1,$

$$z^{1/2} = (\sqrt{13})^{1/2}[\cos(4.4184) + \mathrm{j}\sin(4.4184)] = -0.5502 - \mathrm{j}1.8174$$

Exercise 4.2:

1) Express the following complex numbers in polar form and exponential form.

   a) $4 - j6$

   b) $-7 + j5$

   c) $-10 - j11$

2) Express the following complex numbers in Cartesian form.

   a) $e^{j1.1456}$

   b) $6e^{-3+j2.6524}$

   c) $4e^{2+j0.6728}$

[Ans: $\sqrt{52}(-0.9828), \sqrt{74}(2.5213), \sqrt{221}(-2.3086); 0.4125 + j0.9110, -0.2637 + j0.1404, 23.1153 + j18.4188$]

Exercise 4.2:

3) Evaluate $(2 - j6)^3$ and $(-3 + j4)^7$.

4) Evaluate $(4 - j5)^{1/3}$.

[Ans: $-208.0198 + j143.9714; -76443.2894 + j16122.6278;$

$k = 0\ (1.7747 - j0.5464),$

$k = 1\ (-0.4141 + j1.8101),$

$k = 2\ (-1.3606 - j1.2636)$]