ENGINEERING MATHEMATICS 1

BMFG 1313

EIGENVALUES AND EIGENVECTORS

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Lesson Outcome

Upon completion of this lesson, the student should be able to:

1. Compute all eigenvalues and the corresponding eigenvectors by using Polynomial Method.
2. Compute the dominant eigenvalue and the corresponding eigenvector by using Power Method.

3.1 Solving for Eigenvalues and Eigenvectors

How to find the eigenvalues and eigenvectors?

- By **Polynomial method** or **Power method**

Why we need eigenvalues and eigenvectors?

- ✓ Eigenvalues are used to study differential equations and continuous dynamical systems.
- ✓ They provide critical information in engineering design.
- ✓ Dominant eigenvalues are of primary interest in many physical applications.

3.1 Solving for Eigenvalues and Eigenvectors

An **eigenvector** of an $n \times n$ matrix A is a nonzero vector $\mathbf{v}$ such that $A\mathbf{v} = \lambda\mathbf{v}$ for some scalar λ.

A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution $\mathbf{v}$ of $A\mathbf{v} = \lambda\mathbf{v}$

Nontrivial solution:

At least one of the entries is nonzero

$\mathbf{v} \in \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$ or $\mathbf{v} \in \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$

Trivial solution:

All entries are zero

$$\mathbf{v} \in \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

3.1 Solving for Eigenvalues and Eigenvectors

Example of eigenvalue and eigenvector from $A\mathbf{v} = \lambda\mathbf{v}$

Given $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}, \lambda = -4$

$$\mathrm{A}\mathbf{v} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} -24 \\ -20 \end{bmatrix} = -4 \begin{bmatrix} 6 \\ -5 \end{bmatrix} = \lambda\mathbf{v}$$

Hence, $\mathbf{v} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$ is an eigenvector of *A* corresponding to an eigenvalue $\lambda = -4$.

3.2 Solutions of Linear Systems

There are 3 types of solutions for linear systems:

1) No solution

⇒ Inconsistent linear system,

e.g., $$\begin{bmatrix} 1 & 2 & 3 & \vdots & 2 \\ 0 & 2 & 0 & \vdots & 3 \\ 0 & 0 & 0 & \vdots & \mathbf{5} \end{bmatrix}. \quad (0+0+0 \neq 5)$$

2) solution

⇒ Number of leadings = number of variables (in echelon form),

e.g.,

$$\begin{array}{c} \begin{array}{ccccc} x & y & z & & \end{array} \\ \begin{bmatrix} 1 & 2 & 3 & \vdots & 0 \\ 0 & 1 & 0 & \vdots & 0 \\ 0 & 0 & 1 & \vdots & 0 \end{bmatrix} \end{array} \quad \text{or} \quad \begin{array}{c} \begin{array}{ccccc} x & y & z & & \end{array} \\ \begin{bmatrix} 1 & 2 & 3 & \vdots & 1 \\ 0 & 1 & 2 & \vdots & 2 \\ 0 & 0 & 1 & \vdots & 3 \end{bmatrix} \end{array}$$

3.2 Solutions of Linear Systems

3) Infinitely many solutions

$\Rightarrow$ Number of leadings $<$ Number of variables (in echelon form),

e.g.,

No leading for y and z

$$\begin{array}{c} \begin{array}{cccccc} w & x & y & z & & \end{array} \\ \begin{bmatrix} 1 & 2 & 3 & 4 & \vdots & 1 \\ 0 & 1 & 0 & 0 & \vdots & 2 \\ 0 & 0 & 0 & 0 & \vdots & 0 \end{bmatrix} \end{array} \ (2 < 4)$$

or

No leading for y

$$\begin{array}{c} \begin{array}{ccccc} x & y & z & & \end{array} \\ \begin{bmatrix} 1 & 2 & 3 & \vdots & 1 \\ 0 & 0 & 1 & \vdots & 2 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix} \end{array} \ (2 < 3)$$

Note:
The “staircase” is allowed to move horizontally (to the right) across more than 1 entry but **moves down across only 1 entry.** The entries underneath the “staircase” are all zeroes.

3.2 Solutions of Linear Systems

Example:

Given a linear system which is row reduced as follows:

$$\begin{array}{cccccc} w & x & y & z & & \\ 1 & 1 & 2 & 0 & \vdots & 1 \\ 0 & 1 & 0 & 0 & \vdots & 2 \\ 0 & 0 & 0 & 1 & \vdots & 1 \end{array}$$

Since there is no leading in the column of variable y (or y is free), hence let $y = c, \quad c \in \mathbb{R}$.

By backward substitution,

$$z = 1, \quad x = 2, \quad w + x + 2y = 1 \quad \Rightarrow \quad w = 1 - x - 2y = -1 - 2c$$

Write the solution in the parametric vector form:

$$\therefore \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 - 2c \\ 2 \\ c \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix} + c \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \qquad c \in \mathbb{R}$$

3.2 Solutions of Linear Systems

Exercise 3.1:

Given linear systems which are row reduced as follows:

1) $$\begin{bmatrix} 1 & 2 & 1 & 4 & \vdots & 3 \\ 0 & 1 & 1 & 0 & \vdots & 2 \\ 0 & 0 & 1 & 1 & \vdots & 1 \end{bmatrix}$$

2) $$\begin{bmatrix} 1 & 2 & 3 & 1 & \vdots & 4 \\ 0 & 0 & 1 & 2 & \vdots & 5 \\ 0 & 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$$

Write the solution in the parametric vector form.

[Ans: $\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} -5 \\ 1 \\ -1 \\ 1 \end{bmatrix}, c \in \mathbb{R}; \begin{bmatrix} -11 \\ 0 \\ 5 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ 0 \\ -2 \\ 1 \end{bmatrix}, c_1, c_2 \in \mathbb{R}$]

3.3 Polynomial Method

From the relation of

$$A\mathbf{v} = \lambda\mathbf{v}$$

where λ is one of the eigenvalues of matrix A and $\mathbf{v}$ is the corresponding eigenvector of λ, identity matrix I can be used to express

$$A\mathbf{v} = \lambda I\mathbf{v}$$

Hence,

$$A\mathbf{v} - \lambda I\mathbf{v} = \mathbf{0}$$

and

$$(A - \lambda I)\mathbf{v} = \mathbf{0}$$

To solve this, we need to find λ in such a way $\mathbf{v} \neq \mathbf{0}$.

Thus, to find eigenvalues, we need to solve the **characteristic polynomial**

$$p(\lambda) = |A - \lambda I| = 0$$

3.3 Polynomial Method

Example:

Determine the eigenvalues of the given matrices by using polynomial method and find their corresponding eigenvectors.

1) $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$

2) $B = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$

3.3 Polynomial Method

Solution of Q1:

Step 1: Find the characteristic equation, $|A - \lambda I| = 0$

$$A - \lambda I = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{vmatrix} = (2-\lambda)(-6-\lambda) - 3(3) = 0$$

It gives

$$\lambda^2 + 4\lambda - 21 = 0$$

$$(\lambda - 3)(\lambda + 7) = 0$$

Hence, the eigenvalues are

$$\lambda_1 = 3, \qquad \lambda_2 = -7$$

3.3 Polynomial Method

Solution of Q1:

Step 2: Find the corresponding eigenvector for λ_1, $(A - \lambda I)\mathbf{v} = \mathbf{0}$

$\lambda_1 = 3$: (substitute λ with $\lambda_1 = 3$)

$$(A - 3I)\mathbf{v} = \mathbf{0}$$

$$\left[\begin{array}{cc|c} 2-3 & 3 & 0 \\ 3 & -6-3 & 0 \end{array}\right] = \left[\begin{array}{cc|c} -1 & 3 & 0 \\ 3 & -9 & 0 \end{array}\right] \xrightarrow{3r_1+r_2} \left[\begin{array}{cc|c} -1 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Let $x_2 = c$,

$$-x_1 + 3x_2 = 0$$

$$-x_1 + 3c = 0$$

$$x_1 = 3c$$

Hence,

$$\mathbf{v}_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3c \\ c \end{bmatrix} = c\begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

3.3 Polynomial Method

Solution of Q1:

Step 2.1: Normalize the eigenvector $\mathbf{v}_1$ (largest magnitude of elements is 1)

$$\mathbf{v}_1 = c\begin{bmatrix}3\\1\end{bmatrix}$$

By taking $c = 1$ and normalize the eigenvector so that the largest magnitude of elements is 1:

$$\mathbf{v}_1 = \frac{1}{3}\begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}1\\1/3\end{bmatrix}$$

Hence,

Eigenvalue, $\lambda_1 = 3$ and eigenvector, $\mathbf{v}_1 = \begin{bmatrix}1.0000\\0.3333\end{bmatrix}$

3.3 Polynomial Method

Solution of Q1:

Step 3: Find the corresponding eigenvector for λ_2, $(A-\lambda I)\mathbf{v}=\mathbf{0}$

$\lambda_2=-7$: (substitute λ with $\lambda_2=-7$)

$$(A+7I)\mathbf{v}=\mathbf{0}$$

$$\left[\begin{array}{cc|c} 2-(-7) & 3 & 0 \\ 3 & -6-(-7) & 0 \end{array}\right]=\left[\begin{array}{cc|c} 9 & 3 & 0 \\ 3 & 1 & 0 \end{array}\right]\xrightarrow{-\frac{1}{3}r_1+r_2}\left[\begin{array}{cc|c} 9 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Let $x_2=c$,

$$9x_1+3x_2=0$$

$$9x_1+3c=0\Rightarrow x_1=-\frac{1}{3}c$$

Hence,

$$\mathbf{v}_2=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} -\frac{1}{3}c \\ c \end{bmatrix}=c\begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix}$$

3.3 Polynomial Method

Solution of Q1:

Step 3.1: Normalize the eigenvector $\mathbf{v}_2$ (largest magnitude of elements is 1)

$$\mathbf{v}_2 = c\begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix}$$

By taking $c = 1$ and normalize the eigenvector so that the largest magnitude of elements is 1:

$$\mathbf{v}_2 = \begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix}$$

Hence,

Eigenvalue, $\lambda_2 = -7$ and eigenvector, $\mathbf{v}_2 = \begin{bmatrix} -0.3333 \\ 1.0000 \end{bmatrix}$

3.3 Polynomial Method

Solution of Q2:

Step 1: Find the characteristic equation, $|A-\lambda I|=0$

$$|A-\lambda I|=\begin{vmatrix}1-\lambda & 3 & 3\\ -3 & -5-\lambda & -3\\ 3 & 3 & 1-\lambda\end{vmatrix}$$

$$=(1-\lambda)(-5-\lambda)(1-\lambda)-27-27-[9(-5-\lambda)-9(1-\lambda)-9(1-\lambda)]$$

$$=0$$

It gives

$$-\lambda^3-3\lambda^2+4=0$$

$$-(\lambda-1)(\lambda+2)^2=0$$

Hence, the eigenvalues are

$$\lambda_1=1,\qquad \lambda_2=\lambda_3=-2$$

3.3 Polynomial Method

Solution of Q2:

Step 2: Find the corresponding eigenvector for λ_1, $(A-\lambda I)\mathbf{v}=\mathbf{0}$

$\lambda_1 = 1$:

$$(A-I)\mathbf{v}=\mathbf{0}$$

$$\left[\begin{array}{ccc|c} 0 & 3 & 3 & 0 \\ -3 & -6 & -3 & 0 \\ 3 & 3 & 0 & 0 \end{array}\right] \xrightarrow{r_1 \leftrightarrow r_3} \left[\begin{array}{ccc|c} 3 & 3 & 0 & 0 \\ -3 & -6 & -3 & 0 \\ 0 & 3 & 3 & 0 \end{array}\right] \xrightarrow{r_1+r_2} \left[\begin{array}{ccc|c} 3 & 3 & 0 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 3 & 3 & 0 \end{array}\right] \xrightarrow{-r_2+r_3} \left[\begin{array}{ccc|c} 3 & 3 & 0 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Let $x_3 = c$,

$$-3x_2 - 3x_3 = 0$$
$$-3x_2 - 3c = 0$$
$$x_2 = -c$$

$$3x_1 + 3x_2 = 0$$
$$3x_1 + 3(-c) = 0$$
$$x_1 = c$$

$$\mathbf{v}_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} c \\ -c \\ c \end{bmatrix} = c\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

3.3 Polynomial Method

Solution of Q2:

Step 2.1: Normalize the eigenvector $\mathbf{v}_1$ (largest magnitude of elements is 1)

$$\mathbf{v}_1 = c\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

By taking $c = 1$:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Hence,

Eigenvalue, $\lambda_1 = 1$ and eigenvector, $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

3.3 Polynomial Method

Solution of Q2:

Step 3: Find the corresponding eigenvector for λ_2 *and* λ_3*,* $(A-\lambda I)\mathbf{v}=\mathbf{0}$

$\lambda_2=\lambda_3=-2$:

$$(A+2I)\mathbf{v}=\mathbf{0}$$

$$\left[\begin{array}{ccc|c} 3 & 3 & 3 & 0 \\ -3 & -3 & -3 & 0 \\ 3 & 3 & 3 & 0 \end{array}\right] \xrightarrow[-r_1+r_3]{r_1+r_2} \left[\begin{array}{ccc|c} 3 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Let $x_2=c_1$ and $x_3=c_2$

$$3x_1+3x_2+3x_3=0$$

$$3x_1+3c_1+3c_2=0$$

$$x_1=-c_1-c_2$$

Hence, $$\mathbf{v}=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} -c_1-c_2 \\ c_1 \\ c_2 \end{bmatrix}=c_1\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}+c_2\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

3.3 Polynomial Method

Solution of Q2:

Step 3.1: Normalize the eigenvector $\mathbf{v}$ (largest magnitude of elements is 1)

$$\mathbf{v} = c_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

By taking $c_1 = c_2 = 1$,

$$\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \text{and } \mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Hence,

Eigenvalue, $\lambda_2 = \lambda_3 = -2$ and eigenvector, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

3.3 Polynomial Method

Exercise 3.2:

Determine the eigenvalues of the given matrices by using polynomial method and find their corresponding eigenvalues.

1) $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$

2) $B = \begin{bmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix}$

3) $C = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$

[Ans: $\lambda_1 = 7, \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \lambda_2 = -4, \mathbf{v}_2 = \begin{bmatrix} 1 \\ -0.8333 \end{bmatrix}; \lambda_1 = 3, \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \lambda_2 = 2, \mathbf{v}_2 = \begin{bmatrix} 1 \\ -0.3 \\ -0.1 \end{bmatrix}, \lambda_3 = 0, \mathbf{v}_3 = \begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix};$

$\lambda_1 = 3, \mathbf{v}_1 = \begin{bmatrix} -0.5 \\ 1 \\ -0.5 \end{bmatrix}, \lambda_2 = 1, \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \lambda_3 = 0, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$]

3.4 Power Method (with scaling)

Let $\lambda_1, \lambda_2, \ldots \lambda_n$ be the eigenvalues of an $n \times n$ matrix A. λ_1 is known as a strictly **dominant eigenvalue** of *A* if

$$|\lambda_1| \underset{\substack{\downarrow \\ \text{Strictly larger}}}{>} |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n|$$

(E.g: $\lambda_1 = 3, \lambda_2 = \lambda_3 = 2$ or $\lambda_1 = -5, \lambda_2 = 2, \lambda_3 = 1$)

The eigenvectors corresponding to λ_1 are called dominant eigenvectors of *A*.

3.4 Power Method (with scaling)

Example:

Apply the power method to $A = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix}$ with $\mathbf{v}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Estimate the dominant eigenvalue and a corresponding eigenvector of A accurate to within $\varepsilon = 0.0002$.

3.4 Power Method (with scaling)

Solution:

Step 1: Select an initial vector $\mathbf{v}_0$ *whose largest entry is 1. Then compute* $A\mathbf{v}_k$.

Let $\mathbf{v}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$,

$$A\mathbf{v}_0 = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

Step 2: Let m_k *be an entry in* $A\mathbf{v}_k$ *which gives the highest absolute value.*

$$m_0 = 5$$

3.4 Power Method (with scaling)

Step 3: Compute $\mathbf{v}_{k+1} = \left(\frac{1}{m_k}\right) A\mathbf{v}_k$ *and error.*

$$\mathbf{v}_1 = \left(\frac{1}{m_0}\right) A\mathbf{v}_0 = \frac{1}{5}\begin{bmatrix}5\\2\end{bmatrix} = \begin{bmatrix}1\\0.4\end{bmatrix}$$

$$\|\mathbf{v}_1 - \mathbf{v}_0\|_\infty = 1$$

$$\|\mathbf{v}_1 - \mathbf{v}_0\|_\infty = \left\|\begin{bmatrix}1\\0.4\end{bmatrix} - \begin{bmatrix}0\\1\end{bmatrix}\right\|_\infty = \left\|\begin{bmatrix}1\\-0.6\end{bmatrix}\right\|_\infty$$

$$= \max\{|1|, |-0.6|\} = 1$$

3.4 Power Method (with scaling)

Step 4: Compute $A\mathbf{v}_{k+1}$ *and repeat Step 1-4.*

$$A\mathbf{v}_1 = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.4 \end{bmatrix} = \begin{bmatrix} 8 \\ 1.8 \end{bmatrix}, \qquad m_1 = 8$$

$$\mathbf{v}_2 = \frac{1}{8} \begin{bmatrix} 8 \\ 1.8 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.225 \end{bmatrix},$$

$$||\mathbf{v}_2 - \mathbf{v}_1||_\infty = 0.175$$

3.4 Power Method (with scaling)

$$A\mathbf{v}_2 = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.225 \end{bmatrix} = \begin{bmatrix} 7.125 \\ 1.450 \end{bmatrix}, \qquad m_2 = 7.125$$

$$\mathbf{v}_3 = \frac{1}{7.125} \begin{bmatrix} 7.125 \\ 1.450 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.2035 \end{bmatrix},$$

$$\|\mathbf{v}_3 - \mathbf{v}_2\|_\infty = 0.0215$$

$$A\mathbf{v}_3 = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.2035 \end{bmatrix} = \begin{bmatrix} 7.0175 \\ 1.407 \end{bmatrix}, \qquad m_3 = 7.0175$$

$$\mathbf{v}_4 = \frac{1}{7.0175} \begin{bmatrix} 7.0175 \\ 1.407 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.2005 \end{bmatrix}$$

$$\|\mathbf{v}_4 - \mathbf{v}_3\|_\infty = 0.0030$$

3.4 Power Method (with scaling)

$$A\mathbf{v}_4 = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.2005 \end{bmatrix} = \begin{bmatrix} 7.0025 \\ 1.401 \end{bmatrix}, \qquad m_4 = 7.0025$$

$$\mathbf{v}_5 = \frac{1}{7.0025} \begin{bmatrix} 7.0025 \\ 1.401 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.2001 \end{bmatrix}$$

$$\|\mathbf{v}_5 - \mathbf{v}_4\|_\infty = 0.0004$$

$$A\mathbf{v}_5 = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.2001 \end{bmatrix} = \begin{bmatrix} 7.0005 \\ 1.4002 \end{bmatrix}, \qquad m_5 = 7.0005$$

$$\mathbf{v}_6 = \frac{1}{7.0005} \begin{bmatrix} 7.0005 \\ 1.4002 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.2000 \end{bmatrix}$$

$$\|\mathbf{v}_6 - \mathbf{v}_5\|_\infty = 0.0001$$

3.4 Power Method (with scaling)

$$A\mathbf{v}_6 = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.2000 \end{bmatrix} = \begin{bmatrix} 7.000 \\ 1.4000 \end{bmatrix}, \qquad m_6 = 7.0000$$

Step 5: $\{m_k\}$ approaches the dominant eigenvalue,
$\{\mathbf{v}_k\}$ approaches a corresponding eigenvector.

Dominant eigenvalue $= m_6 = 7$

Dominant eigenvector $= \mathbf{v}_6 = \begin{bmatrix} 1 \\ 0.2000 \end{bmatrix}$

Note: If $|\lambda_2/\lambda_1|$ close to 1, then the power method converges slowly

3.4 Power Method (with scaling)

Example:

Apply the power method to $A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$ with $\mathbf{v}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Estimate the dominant eigenvalue and a corresponding eigenvector of A accurate to within $\varepsilon = 0.001$.

3.4 Power Method (with scaling)

Solution:

$$A\mathbf{v}_0 = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -10 \\ -4 \end{bmatrix}, \qquad m_0 = -10$$

$$\mathbf{v}_1 = \frac{1}{-10}\begin{bmatrix} -10 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.4 \end{bmatrix}$$

$$\|\mathbf{v}_1 - \mathbf{v}_0\|_\infty = 0.6$$

$$A\mathbf{v}_1 = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ 0.4 \end{bmatrix} = \begin{bmatrix} -2.8 \\ -1 \end{bmatrix}, \qquad m_1 = -2.8$$

$$\mathbf{v}_2 = \frac{1}{-2.8}\begin{bmatrix} -2.8 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.3571 \end{bmatrix}$$

$$\|\mathbf{v}_2 - \mathbf{v}_1\|_\infty = 0.0429$$

3.4 Power Method (with scaling)

Solution:

$$A\mathbf{v}_2 = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 1 \\ 0.3571 \end{bmatrix} = \begin{bmatrix} -2.2852 \\ -0.7855 \end{bmatrix}, \qquad m_2 = -2.2852$$

$$\mathbf{v}_3 = \frac{1}{-2.2852} \begin{bmatrix} -2.2852 \\ -0.7855 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.3437 \end{bmatrix}$$

$$\|\mathbf{v}_3 - \mathbf{v}_2\|_\infty = 0.0134$$

$$A\mathbf{v}_3 = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 1 \\ 0.3437 \end{bmatrix} = \begin{bmatrix} -2.1244 \\ -0.7185 \end{bmatrix}, \qquad m_3 = -2.1244$$

$$\mathbf{v}_4 = \frac{1}{-2.1244} \begin{bmatrix} -2.1244 \\ -0.7185 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.3382 \end{bmatrix}$$

$$\|\mathbf{v}_4 - \mathbf{v}_3\|_\infty = 0.0055$$

3.4 Power Method (with scaling)

Solution:

$$A\mathbf{v}_4 = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 1 \\ 0.3382 \end{bmatrix} = \begin{bmatrix} -2.0584 \\ -0.6910 \end{bmatrix}, \qquad m_4 = -2.0584$$

$$\mathbf{v}_5 = \frac{1}{-2.0584} \begin{bmatrix} -2.0584 \\ -0.6910 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.3357 \end{bmatrix}$$

$$\|\mathbf{v}_5 - \mathbf{v}_4\|_\infty = 0.0025$$

$$A\mathbf{v}_5 = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 1 \\ 0.3357 \end{bmatrix} = \begin{bmatrix} -2.0284 \\ -0.6785 \end{bmatrix}, \qquad m_5 = -2.0284$$

$$\mathbf{v}_6 = \frac{1}{-2.0284} \begin{bmatrix} -2.0284 \\ -0.6785 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.3345 \end{bmatrix}$$

$$\|\mathbf{v}_6 - \mathbf{v}_5\|_\infty = 0.0012$$

3.4 Power Method (with scaling)

Solution:

$$A\mathbf{v}_6 = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 1 \\ 0.3345 \end{bmatrix} = \begin{bmatrix} -2.0140 \\ -0.6725 \end{bmatrix}, \qquad m_6 = -2.0140$$

$$\mathbf{v}_7 = \frac{1}{-2.0140} \begin{bmatrix} -2.0140 \\ -0.6725 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.3339 \end{bmatrix}$$

$$\|\mathbf{v}_7 - \mathbf{v}_6\|_\infty = 0.0006 < 0.001$$

$$A\mathbf{v}_7 = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 1 \\ 0.3339 \end{bmatrix} = \begin{bmatrix} -2.0068 \\ -0.6695 \end{bmatrix}, \qquad m_7 = -2.0068$$

Dominant eigenvalue $= m_7 = -2.0068$

Dominant eigenvector $= \mathbf{v}_7 = \begin{bmatrix} 1 \\ 0.3339 \end{bmatrix}$

3.4 Power Method (with scaling)

Exercise 3.2:

Apply the power method to $A = \begin{bmatrix} -1 & -6 & 0 \\ 2 & 7 & 0 \\ 1 & 2 & -1 \end{bmatrix}$ with $\mathbf{v}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Estimate the dominant eigenvalue and a corresponding eigenvector of A accurate to within $\varepsilon = 0.005$.

[Ans: $m_5 = 5.0006$, $\mathbf{v}_5 = \begin{bmatrix} -0.9997 \\ 1 \\ 0.1668 \end{bmatrix}$]