# BEKG 2452
# NUMERICAL METHODS
# Solution of Linear Systems (Gauss-Seidel & Application)

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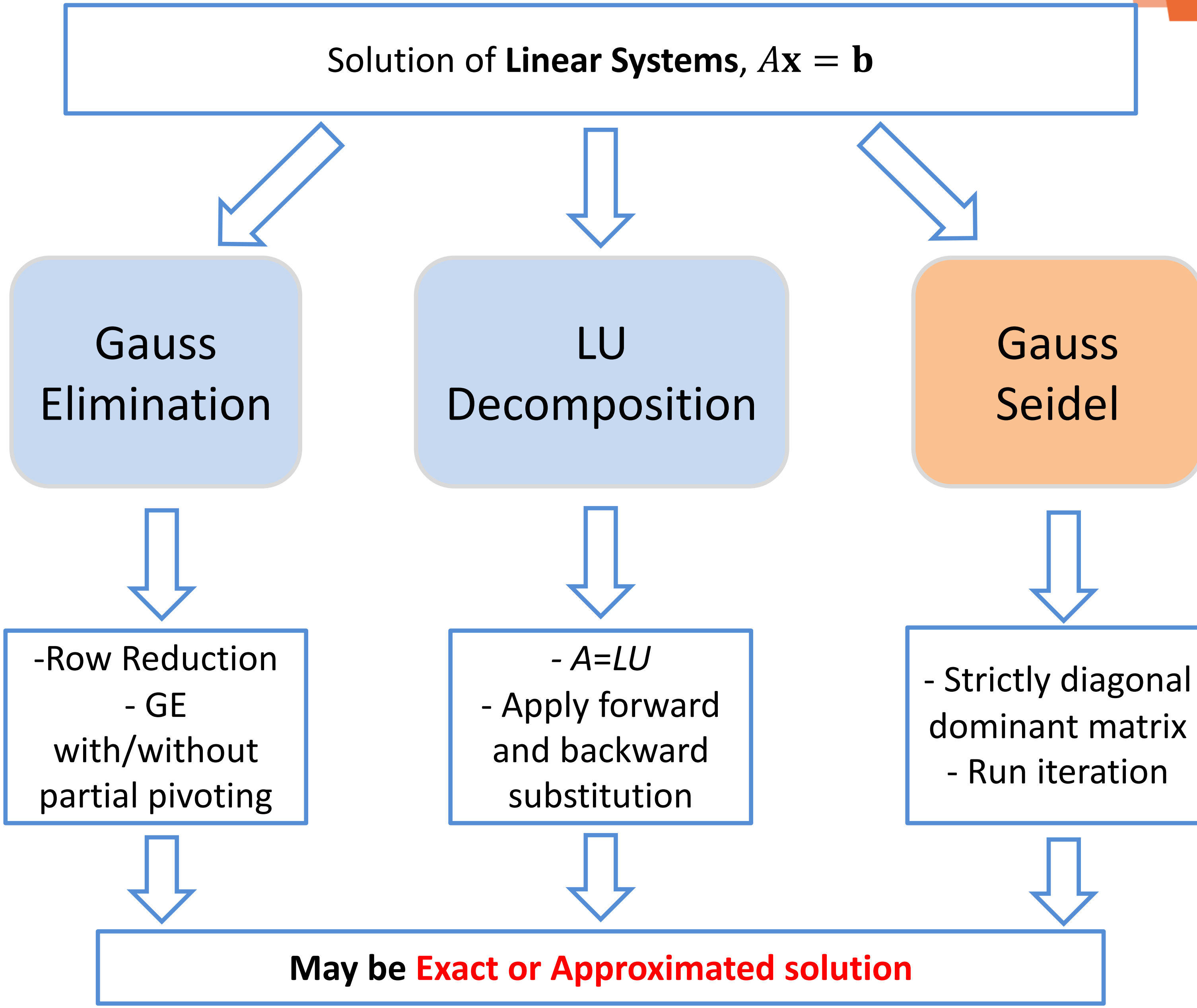
# Lesson Outcome

Upon completion of this lesson, the student should be able to:

1. Solve a linear system by using Gauss Seidel.
2. Compute the dominant eigenvalue and the corresponding eigenvector by using Power Method.


Solution of **Linear Systems**, $A\mathbf{x} = \mathbf{b}$
Gauss Elimination
LU Decomposition
Gauss Seidel
-Row Reduction
- GE with/without partial pivoting
- *A=LU*
- Apply forward and backward substitution
- Strictly diagonal dominant matrix
- Run iteration
**May be Exact or Approximated solution**

## 3.3 Gauss Seidel

### Strictly Diagonal Dominant Matrix

- The **magnitude of diagonal entry** in a row is larger than the sum of the magnitudes of all the other entries in that row:

$$|a_{kk}| > \sum_{j \neq k}^{N} |a_{kj}| \text{ for } k = 1,2, \dots, N)$$

e.g.

$$\begin{bmatrix} 6 & 2 & 3 \\ -1 & -9 & 4 \\ 3 & 0 & -7 \end{bmatrix}$$

$|6| > |2| + |3|$

$|-9| > |-1| + |4|$

$|-7| > |3| + |0|$

# 3.3 Gauss Seidel

*Basic Idea:*

Solve unknown variable of a linear system iteratively by using previously computed results as soon as they are available.

*Flow:*

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

Linear System

Strictly diagonal dominant matrix arrangement

Form the iterative sequence

$$\left(x_1^{(k+1)}, x_2^{(k+1)}, x_3^{(k+1)}\right)$$

Initial guess: $(x_1^{(0)}, x_2^{(0)}, x_3^{(0)})$

Stop iteration when

$$\left\|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\right\|_\infty = \max_{1\le i\le n} \left\{\left|x_i^{(k)} - x_i^{(k-1)}\right|\right\} < \varepsilon$$

(known as infinity norm)

## 3.3.1 Gauss Seidel Iterative Sequence

From $A\mathbf{x} = \mathbf{b}$ (A is a strictly diagonal dominant matrix),

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

$$\Downarrow$$

$$x_1^{(k+1)} = \frac{1}{a_{11}}\left(b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)}\right)$$

$$x_2^{(k+1)} = \frac{1}{a_{22}}\left(b_2 - a_{21}x_1^{(k+1)} - a_{23}x_3^{(k)}\right)$$

$$x_3^{(k+1)} = \frac{1}{a_{33}}\left(b_3 - a_{31}x_1^{(k+1)} - a_{32}x_2^{(k+1)}\right)$$

## Example 3.12:

Solve the following linear system by using Gauss Seidel.

$$\begin{aligned} 12x_1 + 3x_2 - x_3 &= 15 \\ 2x_1 - x_2 + 10x_3 &= 30 \\ x_1 + 8x_2 + x_3 &= 20 \end{aligned}$$

Start the initial guess with $\mathbf{x}^{(0)} = \mathbf{0}$ and stop the iteration when $\left\|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\right\|_\infty < 0.001$

## Solution:

Check for strictly diagonal dominant:

$$\begin{aligned} 12x_1 + 3x_2 - x_3 &= 15 \\ 2x_1 - x_2 + 10x_3 &= 30 \\ x_1 + 8x_2 + x_3 &= 20 \end{aligned} \xrightarrow{r_2 \leftrightarrow r_3} \begin{aligned} 12x_1 + 3x_2 - x_3 &= 15 \\ x_1 + 8x_2 + x_3 &= 20 \\ 2x_1 - x_2 + 10x_3 &= 30 \end{aligned}$$

Iterative Sequence:

$$x_1^{(k+1)} = \frac{1}{12}\left(15 - 3x_2^{(k)} + x_3^{(k)}\right)$$

$$x_2^{(k+1)} = \frac{1}{8}\left(20 - x_1^{(k+1)} - x_3^{(k)}\right)$$

$$x_3^{(k+1)} = \frac{1}{10}\left(30 - 2x_1^{(k+1)} + x_2^{(k+1)}\right)$$

## Solution:

$$\max_{1\le i\le n}\left\{\left|x_i^{(k)}-x_i^{(k-1)}\right|\right\}$$

| $k$ | $x_1^{(k)}$ | $x_2^{(k)}$ | $x_3^{(k)}$ | $\left\Vert \mathbf{x}^{(k)}-\mathbf{x}^{(k-1)}\right\Vert_\infty$ |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | |
| 1 | 1.2500 | 2.3438 | 2.9844 | 2.9844 |
| 2 | 0.9128 | 2.0129 | 3.0187 | 0.0343 |
| 3 | 0.9983 | 1.9979 | 3.0001 | 0.0855 |
| 4 | 1.0005 | 1.9999 | 2.9999 | 0.0022 |
| 5 | 1.0000 | 2.0000 | 3.0000 | 0.0005 |

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

## Example 3.13:

Solve the following linear system by using Gauss Seidel.

1) $$0.3x_1 - 0.2x_2 + 10x_3 = 71.4$$
$$3x_1 - 0.1x_2 - 0.2x_3 = 7.85$$
$$0.1x_1 + 7x_2 - 0.3x_3 = -19.3$$

2) $$-x_1 + x_2 + 7x_3 = -6$$
$$4x_1 - x_2 - x_3 = 3$$
$$-2x_1 + 6x_2 + x_3 = 9$$

Start the initial guess with $\mathbf{x}^{(0)} = \mathbf{0}$ and stop the iteration when $\left\|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\right\|_\infty < 0.0005$

# Solution: Q1

Check for strictly diagonal dominant:

$$3x_1 - 0.1x_2 - 0.2x_3 = 7.85$$
$$0.1x_1 + 7x_2 - 0.3x_3 = -19.3$$
$$0.3x_1 - 0.2x_2 + 10x_3 = 71.4$$

Iterative Sequence:

$$x_1^{(k+1)} = \frac{1}{3}\left(7.85 + 0.1x_2^{(k)} + 0.2x_3^{(k)}\right)$$
$$x_2^{(k+1)} = \frac{1}{7}\left(-19.3 - 0.1x_1^{(k+1)} + 0.3x_3^{(k)}\right)$$
$$x_3^{(k+1)} = \frac{1}{10}\left(71.4 - 0.3x_1^{(k+1)} + 0.2x_2^{(k+1)}\right)$$

## Solution: Q1

$$\max_{1\le i\le n}\left\{\left|x_i^{(k)} - x_i^{(k-1)}\right|\right\}$$

| $k$ | $x_1^{(k)}$ | $x_2^{(k)}$ | $x_3^{(k)}$ | $\left\Vert \mathbf{x}^{(k)} - \mathbf{x}^{(k-1)} \right\Vert_\infty$ |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | |
| 1 | 2.6167 | -2.7945 | 7.0056 | 7.0056 |
| 2 | 2.9906 | -2.4996 | 7.0003 | 0.3739 |
| 3 | 3.0000 | -2.5000 | 7.000 | 0.0094 |
| 4 | 3.0000 | -2.5000 | 7.0000 | 0.0000 |

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -2.5 \\ 7 \end{bmatrix}$$

# Solution: Q2

Check for strictly diagonal dominant:

$$4x_1 - x_2 - x_3 = 3$$
$$-2x_1 + 6x_2 + x_3 = 9$$
$$-x_1 + x_2 + 7x_3 = -6$$

Iterative Sequence:

$$x_1^{(k+1)} = \frac{1}{4}\left(3 + x_2^{(k)} + x_3^{(k)}\right)$$

$$x_2^{(k+1)} = \frac{1}{6}\left(9 + 2x_1^{(k+1)} - x_3^{(k)}\right)$$

$$x_3^{(k+1)} = \frac{1}{7}\left(-6 + x_1^{(k+1)} - x_2^{(k+1)}\right)$$

# Solution: Q2

$$\max_{1 \le i \le n} \left\{ x_i^{(k)} - x_i^{(k-1)} \right\}$$

| $k$ | $x_1^{(k)}$ | $x_2^{(k)}$ | $x_3^{(k)}$ | $\left\| \mathbf{x}^{(k)} - \mathbf{x}^{(k-1)} \right\|_\infty$ |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | |
| 1 | 0.7500 | 1.7500 | -1.0000 | 1.7500 |
| 2 | 0.9375 | 1.9792 | -1.0060 | 0.2292 |
| 3 | 0.9933 | 1.9988 | -1.0008 | 0.0558 |
| 4 | 0.9995 | 2.0000 | -1.0000 | 0.0062 |
| 5 | 1.0000 | 2.0000 | -1.0000 | 0.0005 |

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

# 3.4 Solving for Eigenvalues and Eigenvectors

How to find the eigenvalues and eigenvectors?

- By **Power method**

Why we need eigenvalues and eigenvectors?

- Eigenvalues are used to study differential equations and continuous dynamical systems.
- They provide critical information in engineering design.
- Dominant eigenvalues are of primary interest in many physical applications.

# 3.4 Solving for Eigenvalues and Eigenvectors

An **eigenvector** of an $n \times n$ matrix $A$ is a nonzero vector $\mathbf{v}$ such that $A\mathbf{v} = \lambda\mathbf{v}$ for some scalar $\lambda$.

A scalar $\lambda$ is called an **eigenvalue** of $A$ if there is a nontrivial solution $\mathbf{v}$ of $A\mathbf{v} = \lambda\mathbf{v}$

Nontrivial solution:
At least one of the entries is nonzero

$$\mathbf{v} \in \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \text{ or } \mathbf{v} \in \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

Trivial solution:
All entries are zero

$$\mathbf{v} \in \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

## 3.4 Solving for Eigenvalues and Eigenvectors

### Example of eigenvalue and eigenvector from $A\mathbf{v} = \lambda\mathbf{v}$

Given $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}, \lambda = -4$

$$\mathrm{A}\mathbf{v} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} -24 \\ -20 \end{bmatrix} = -4 \begin{bmatrix} 6 \\ -5 \end{bmatrix} = \lambda\mathbf{v}$$

Hence, $\mathbf{v} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$ is an eigenvector of *A* corresponding to an eigenvalue $\lambda = -4$.

## 3.4.1 Power Method (with scaling)

Let $\lambda_1, \lambda_2, \dots \lambda_n$ be the eigenvalues of an $n \times n$ matrix $A$. $\lambda_1$ is known as a strictly **dominant eigenvalue** of $A$ if

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n|$$

Strictly larger

(E.g: $\lambda_1 = 3, \lambda_2 = \lambda_3 = 2$ or $\lambda_1 = -5, \lambda_2 = 2, \lambda_3 = 1$)

The eigenvectors corresponding to $\lambda_1$ are called dominant eigenvectors of $A$.

## 3.4.1 Power Method (with scaling)

### Example 3.14:

Apply the power method to $A = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix}$ with $\mathbf{v}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Estimate the dominant eigenvalue and a corresponding eigenvector of $A$ accurate to within $\varepsilon = 0.0002$.

### 3.4.1 Power Method (with scaling)

*Step 1: Select an initial vector* $\mathbf{v}_0$ *whose largest entry is 1. Then compute* $A\mathbf{v}_k$*.*

Let $\mathbf{v}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$,

$$A\mathbf{v}_0 = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

*Step 2: Let* $m_k$ *be an entry in* $A\mathbf{v}_k$ *which gives the highest absolute value.*

$$m_0 = 5$$

## 3.4.1 Power Method (with scaling)

*Step 3: Compute* $\mathbf{v}_{k+1} = \left(\frac{1}{m_k}\right) A\mathbf{v}_k$ *and error.*

$$\mathbf{v}_1 = \left(\frac{1}{m_0}\right) A\mathbf{v}_0 = \frac{1}{5}\begin{bmatrix}5\\2\end{bmatrix} = \begin{bmatrix}1\\0.4\end{bmatrix}$$

$$\|\mathbf{v}_1 - \mathbf{v}_0\|_\infty = 1$$

$$\|\mathbf{v}_1 - \mathbf{v}_0\|_\infty = \left\|\begin{bmatrix}1\\0.4\end{bmatrix} - \begin{bmatrix}0\\1\end{bmatrix}\right\|_\infty = \left\|\begin{bmatrix}1\\-0.6\end{bmatrix}\right\|_\infty$$

$$= \max\{|1|, |-0.6|\} = 1$$

## 3.4.1 Power Method (with scaling)

*Step 4: Compute $A\mathbf{v}_{k+1}$ and repeat Step 1-4.*

$$A\mathbf{v}_1 = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.4 \end{bmatrix} = \begin{bmatrix} 8 \\ 1.8 \end{bmatrix}, \qquad m_1 = 8$$

$$\mathbf{v}_2 = \frac{1}{8} \begin{bmatrix} 8 \\ 1.8 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.225 \end{bmatrix},$$

$$\|\mathbf{v}_2 - \mathbf{v}_1\|_\infty = 0.175$$

## 3.4.1 Power Method (with scaling)

$$A\mathbf{v}_2 = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.225 \end{bmatrix} = \begin{bmatrix} 7.125 \\ 1.450 \end{bmatrix}, \qquad m_2 = 7.125$$

$$\mathbf{v}_3 = \frac{1}{7.125} \begin{bmatrix} 7.125 \\ 1.450 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.2035 \end{bmatrix},$$

$$\|\mathbf{v}_3 - \mathbf{v}_2\|_\infty = 0.0215$$

$$A\mathbf{v}_3 = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.2035 \end{bmatrix} = \begin{bmatrix} 7.0175 \\ 1.407 \end{bmatrix}, \qquad m_3 = 7.0175$$

$$\mathbf{v}_4 = \frac{1}{7.0175} \begin{bmatrix} 7.0175 \\ 1.407 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.2005 \end{bmatrix}$$

$$\|\mathbf{v}_4 - \mathbf{v}_3\|_\infty = 0.0030$$

# 3.4.1 Power Method (with scaling)

$$A\mathbf{v}_4 = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.2005 \end{bmatrix} = \begin{bmatrix} 7.0025 \\ 1.401 \end{bmatrix}, \qquad m_4 = 7.0025$$

$$\mathbf{v}_5 = \frac{1}{7.0025} \begin{bmatrix} 7.0025 \\ 1.401 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.2001 \end{bmatrix}$$

$$\|\mathbf{v}_5 - \mathbf{v}_4\|_\infty = 0.0004$$

$$A\mathbf{v}_5 = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.2001 \end{bmatrix} = \begin{bmatrix} 7.0005 \\ 1.4002 \end{bmatrix}, \qquad m_5 = 7.0005$$

$$\mathbf{v}_6 = \frac{1}{7.0005} \begin{bmatrix} 7.0005 \\ 1.4002 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.2000 \end{bmatrix}$$

$$\|\mathbf{v}_6 - \mathbf{v}_5\|_\infty = 0.0001$$

## 3.4.1 Power Method (with scaling)

$$A\mathbf{v}_6 = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.2000 \end{bmatrix} = \begin{bmatrix} 7.000 \\ 1.4000 \end{bmatrix}, \qquad m_6 = 7.0000$$

*Step 5: $\{m_k\}$ approaches the dominant eigenvalue, $\{\mathbf{v}_k\}$ approaches a corresponding eigenvector.*

Dominant eigenvalue $= m_6 = 7$

Dominant eigenvector $= \mathbf{v}_6 = \begin{bmatrix} 1 \\ 0.2000 \end{bmatrix}$

**Note:** If $|\lambda_2/\lambda_1|$ close to 1, then the power method converges slowly

## 3.4.1 Power Method (with scaling)

### Example 3.15:

Apply the power method to $A = \begin{bmatrix} -1 & -6 & 0 \\ 2 & 7 & 0 \\ 1 & 2 & -1 \end{bmatrix}$

with $\mathbf{v}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Estimate the dominant eigenvalue and a corresponding eigenvector of $A$ accurate to within $\varepsilon = 0.005$.

## Solution:

$$A\mathbf{v}_0 = \begin{bmatrix} -1 & -6 & 0 \\ 2 & 7 & 0 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -7 \\ 9 \\ 2 \end{bmatrix}, \qquad m_0 = 9$$

$$\mathbf{v}_1 = \frac{1}{9} \begin{bmatrix} -7 \\ 9 \\ 2 \end{bmatrix} = \begin{bmatrix} -0.7778 \\ 1 \\ 0.2222 \end{bmatrix}$$

$$\|\mathbf{v}_1 - \mathbf{v}_0\|_\infty = 1.7778$$

$$A\mathbf{v}_1 = \begin{bmatrix} -1 & -6 & 0 \\ 2 & 7 & 0 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} -0.7778 \\ 1 \\ 0.2222 \end{bmatrix} = \begin{bmatrix} -5.2222 \\ 5.4444 \\ 1 \end{bmatrix}, m_1 = 5.4444$$

$$\mathbf{v}_2 = \frac{1}{5.4444} \begin{bmatrix} -5.2222 \\ 5.4444 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.9592 \\ 1 \\ 0.1837 \end{bmatrix}$$

$$\|\mathbf{v}_2 - \mathbf{v}_1\|_\infty = 0.1814$$

## Solution:

$$A\mathbf{v}_2 = \begin{bmatrix} -1 & -6 & 0 \\ 2 & 7 & 0 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} -0.9592 \\ 1 \\ 0.1837 \end{bmatrix} = \begin{bmatrix} -5.0408 \\ 5.0816 \\ 0.8571 \end{bmatrix}, m_2 = 5.0816$$

$$\mathbf{v}_3 = \frac{1}{5.0816} \begin{bmatrix} -5.0408 \\ 5.0816 \\ 0.8571 \end{bmatrix} = \begin{bmatrix} -0.9920 \\ 1 \\ 0.1687 \end{bmatrix}$$

$$\|\mathbf{v}_3 - \mathbf{v}_2\|_\infty = 0.0328$$

$$A\mathbf{v}_3 = \begin{bmatrix} -1 & -6 & 0 \\ 2 & 7 & 0 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} -0.9920 \\ 1 \\ 0.1687 \end{bmatrix} = \begin{bmatrix} -5.008 \\ 5.016 \\ 0.8393 \end{bmatrix}, m_3 = 5.016$$

$$\mathbf{v}_4 = \frac{1}{5.016} \begin{bmatrix} -5.008 \\ 5.016 \\ 0.8393 \end{bmatrix} = \begin{bmatrix} -0.9984 \\ 1 \\ 0.1673 \end{bmatrix}$$

$$\|\mathbf{v}_4 - \mathbf{v}_3\|_\infty = 0.0064$$

## Solution:

$$A\mathbf{v}_4 = \begin{bmatrix} -1 & -6 & 0 \\ 2 & 7 & 0 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} -0.9984 \\ 1 \\ 0.1673 \end{bmatrix} = \begin{bmatrix} -5.0016 \\ 5.0032 \\ 0.8343 \end{bmatrix}, m_4 = 5.0032$$

$$\mathbf{v}_5 = \frac{1}{5.0032} \begin{bmatrix} -5.0016 \\ 5.0032 \\ 0.8343 \end{bmatrix} = \begin{bmatrix} -0.9997 \\ 1 \\ 0.1668 \end{bmatrix}$$

$$\|\mathbf{v}_5 - \mathbf{v}_4\|_\infty = 0.0013$$

$$A\mathbf{v}_5 = \begin{bmatrix} -1 & -6 & 0 \\ 2 & 7 & 0 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} -0.9997 \\ 1 \\ 0.1668 \end{bmatrix} = \begin{bmatrix} -5.0003 \\ 5.0006 \\ 0.8335 \end{bmatrix}, m_5 = 5.0006$$

Dominant eigenvalue $= m_5 = 5.0006$

Dominant eigenvector $= \mathbf{v}_5 = \begin{bmatrix} -0.9997 \\ 1 \\ 0.1668 \end{bmatrix}$